Spring 2018 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed/ 60 Minutes

1. Hezzy runs his first lap in 80 seconds, and, since he gets more and more tired as he runs, each following lap takes him six seconds longer than the previous one. If five laps of this track equals a mile, then Hezzy will take how many seconds to run a mile?

2. One counter is in a pouch. It is with equal probability either black or white. A white counter is added to the pouch. Next, one counter is randomly drawn from the pouch and it turns out to be white. What is the probability that the remaining counter in the pouch is black?

3. Let x and y satisfy the system
$$\tan x + \tan y = 20$$
$$\cot x + \cot y = 15$$
Find the value of $\tan(x + y)$.

4. If r and s are the roots of the quadratic equation
$$2x^2 + 2x - 17 = 0$$
find the value of
$$r^2s^2 + rs^2 + sr^2 + rs + 1$$

5. Solve for a:
$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$
Note that the vertical bars here indicate the determinant is to be taken.

6. How many pairs of ordered integers (m, n) satisfy $m \geq 0$, $n \geq 0$, and
$$\frac{m}{144} + \frac{n}{108} \leq 1$$
?

7. A biased coin is such that when tossed eight times the probability of getting exactly three heads is the same as the probability of getting exactly four heads. When this coin is tossed once, what is the probability of getting heads? Assume that the probability of getting heads when tossed once is neither zero nor one.

8. Find one positive integer value of n so that $12^n - 1$ is divisible by 25.

9. Find all ordered triples (x, y, z) that solve the system
$$xy = 5x + 6y - 4z$$
$$y^2 = 3x + 5y - z$$
$$yz = x + 4y + 2z$$

10. A fair coin is tossed 13 times. Find the expected number of HH pairs. We count HH pairs thus—the sequence

$$HTTHHTHHHTTHH$$

has four HH pairs.

11. How many 3×3 matrices with non-negative integer entries have each row sum and each column sum equal to 2?

12. How many ordered pairs of integers (x, y) satisfy

$$x + y = x^2 - xy + y^2$$

?

13. After expanding and simplifying the product

$$a(a+b)(a+b+c)(a+b+c+d)(a+b+c+d+e)\cdots(a+b+c+\cdots+m)$$

how many terms remain?

14. Integers a and b have the property that the cubic equations

$$x^3 + 10x^2 - 16x + a = 0$$
$$x^3 + 18x^2 + 88x + b = 0$$

share exactly two real roots. Find the value of $a + b$.

15. Recall that $\binom{n}{k}$ is the number of ways of choosing k objects from a set of n objects. Find the value of

$$\binom{2019}{0} - \binom{2018}{1} + \binom{2017}{2} - \binom{2016}{3} + \binom{2015}{4} - \cdots + \binom{1011}{1008} - \binom{1010}{1009}$$