Fall 2014 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. A number of students pitch in to buy a gift for their teacher. If each pays 8 dollars, the total collected would be too great by 3 dollars. If each pays 7 dollars, the total collected would be too little by 4 dollars. How much does the gift cost?

2. Write down in order from least to greatest (separate by commas) these irrational numbers:

$$1+\sqrt{3}, \qquad 2-\sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

3. Solve the system

$$\begin{cases} \dfrac{x}{6}+\dfrac{4}{y} & =2 \\ \dfrac{18}{x}+\dfrac{y}{2} & =5 \end{cases}$$

4. In right $\triangle ABC$, its inscribed circle meets legs BA and BC at points D and E respectively. If $BD = 3$ and $DA = 11$, find the length of leg BC.

5. Find one positive value of b so that $x = \tan^{-1} b$ solves the equation

$$\tan^2(2x) + \tan^2(x) = 10$$

6. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?

7. Let the function y satisfy

$$\frac{dy}{dx} = y^2(y-2)(y+2)$$

   If $y(0) = -1/5$ find the limit as $x \to \infty$ of $y(x)$.

8. Let

$$f(x) = \left(\sum_{k=0}^{100} x^k\right) \cdot \left(\sum_{k=0}^{100} (-1)^k x^k\right)$$

   Determine the value of $f'(1)$.

9. Let f be differentiable for all x. Let $f(1) = 1/2$. Suppose for all $x \neq 0$ that $f(x) = f(1/x)$. Let $g(x) = x^4 f(x)$. Find the value of $g'(1)$.

10. Let a be a positive constant. If the minimum value of $f(x) = e^x + ae^{-x}$ is $3a$, find the value of a.

11. Determine
$$\lim_{x \to 0} (\cos x)^{\cot^2 x}$$

12. The infinite expression
$$\sqrt{9 + \sqrt{9 + \sqrt{9 + \cdots}}}$$
can be written in the form $\dfrac{a + \sqrt{b}}{c}$ where a, b, and c are positive integers with no common factor greater than one. Find the value of $a + b + c$.

13. Find the equation of a line which is tangent to $y = x^4 - 8x^2 + 3x + 5$ at two distinct points.

14. Let
$$f(x) = \begin{cases} e^{3x-(1/x^2)} + 2e^{3x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$
Determine the value of $f'(0)$.

15. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 13$.