Spring 2019 McNabb GDCTM Contest
Geometry

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. Solve the equation
$$x + \frac{x}{3} + \frac{x}{5} = 0$$

2. If Hezy drives for three hours at 30 miles per hour followed by five hours at 46 miles per hour, what is his average speed in miles per hour for the whole trip?

3. Simplify $\dfrac{30! - 29!}{31! - 30!}$.

4. A piece of wire is wrapped in a spiral around a cylinder of radius $3/\pi$ and height 12 in such a way that it completes two revolutions as it goes from bottom to top of the cylinder. How long is the wire?

5. The angle bisectors of the angles of $\triangle ABC$ meet at point D. If $\angle A = 100°$, find the measure of $\angle BDC$ in degrees.

6. A group of students plan to contribute equally to a charity. They have decided in advance the total amount of their contribution. If three students drop out of this plan, the remaining students would have to each contribute two dollars more to preserve the original gift. If six of the original students were to drop out of this plan, the remaining students would each have to contribute five dollars more to preserve the original gift. How many students are in the original group?

7. Find the number of ways to color the edges of a square if four colors are available and two colorings are considered the same if one can be rotated into the other.

8. In $\triangle ABC$ points D and E are on sides $\overline{BC}$ and $\overline{AC}$ respectively. Draw segments $\overline{AD}$ and $\overline{BE}$ intersecting at point F. If $AE/EC = 3$ and $BF/FE = 8/3$, then find the value of AF/FD.

9. What is the number of subsets of $\{1, 2, 3, 4, \cdots, 99, 100\}$ with an even number of even numbers? Recall that 0 is an even number. Answer in the form 2^n where n is an integer.

10. Eight spheres, each of radius one, are situated such that their centers form the vertices of a cube and each sphere is externally tangent to exactly three of the other spheres. A ninth sphere is externally tangent to all of the original eight spheres. Find the radius of this ninth sphere.

11. Reflection across the line $y = x/\sqrt{3}$ followed by reflection across the line $y = x\sqrt{3}$ is equivalent to a counter-clockwise rotation of θ degrees, where $0 < \theta < 180$. Find the value of θ.

12. How many paths are there from the point $(0, 0, 0, 0)$ to the point $(2, 2, 2, 2)$ if the only possible moves are to increase just a single coordinate by 1?

13. Two medians of a triangle both have length 9. If the area of the triangle is $24\sqrt{5}$, find the largest possible length of its third median.

14. Given $\triangle ABC$ with area 13, points D, E, and F are located on sides BC, CA, and AB respectively in such a way that $BD/DC = CE/EA = AF/FB = 3/1$. Segments AD, BE, and CF are drawn intersecting each other pairwise in three points G, H, and I. Find the area of $\triangle GHI$.

15. Define the sum of two points $P(a, b)$ and $Q(c, d)$ in the plane to be as if they were vectors, namely, $P + Q = (a + c, b + d)$. Let S and T be two regions in the plane. Define their sum as

$$S + T = \{P + Q \mid P \in S,\ Q \in T\}$$

Let

$$S = \{(x, y) \mid x^2 + y^2 \leq 1\}$$
$$T = \{(x, y) \mid 4 \leq x \leq 5 \quad \text{and} \quad 0 \leq y \leq 1\}$$

Find the area of $S + T$.