Fall 2010 McNabb GDCTM Contest Calculus

NO Calculators Allowed

1. How many non-congruent scalene triangles with integer side lengths exist with two sides of lengths 13 and 7 respectively?

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

2. Let A and B satisfy $\log_2(\log_4 A) = 1$ and $\log_4(\log_2 B) = 1$. Find the value of $\dfrac{1}{A} + \dfrac{1}{B}$.

   (A) 2 **(B)** 1 **(C)** $\dfrac{1}{3}$ **(D)** $\dfrac{1}{8}$ **(E)** $\dfrac{1}{16}$

3. The value of $\sin 195^\circ + \sin 105^\circ$ is equal to

   (A) $\dfrac{1}{2}$ **(B)** $\dfrac{1}{\sqrt{2}}$ **(C)** $\dfrac{\sqrt{3}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

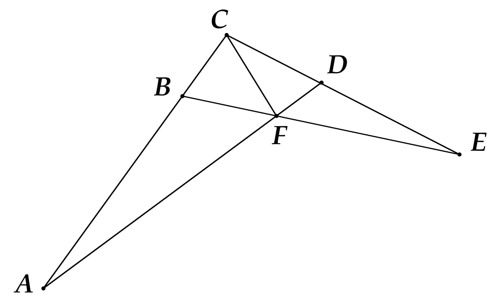
4. A positive integer has the interesting property that when expressed as a three digit base-7 number, those digits are the reverse of its digits when expressed as a base-9 number. What is this number expressed in normal form as a base-10 number?

   (A) 124 **(B)** 241 **(C)** 248 **(D)** 428 **(E)** 503

5. In how many ways can the the letters in the string $ABECEDA$ be arranged so that the consonants are in alphabetical order?

   (A) 90 **(B)** 105 **(C)** 120 **(D)** 180 **(E)** 210

6. In the configuration shown, the area of $\triangle ABF$ is 11, the area of $\triangle CFD$ is 3, and the area of $\triangle DEF$ is $11/3$. Find the area of $\triangle BCF$.



(A) $\dfrac{11}{4}$ **(B)** 3 **(C)** 4 **(D)** $\dfrac{14}{3}$ **(E)** 8

7. Hezy and Zeke were employed at different daily wages. At the end of a certain number of days Hezy received \$300, while Zeke, who had been absent from work two of those days, received only \$192. However, had it been the other way around, had Zeke worked all those days and Hezy been absent twice, then both would have received the same amount. What was Hezy's daily wage?

(A) 30 **(B)** 40 **(C)** 50 **(D)** 60 **(E)** 70

8. What is the slope of the line that bisects the acute angle formed by the lines $y = (5/12)x$ and $y = (3/4)x$?

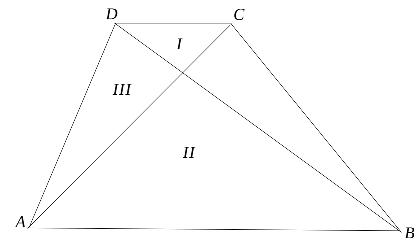
(A) $\dfrac{1}{2}$ **(B)** $\dfrac{7}{12}$ **(C)** $\dfrac{5}{8}$ **(D)** $\dfrac{4}{7}$ **(E)** $\dfrac{2}{3}$

9. The sum of two of the roots of $p(x) = 4x^3 + 8x^2 - 9x - k$, where k is a constant, is zero. Find the value of k.

(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 200

10. An urn contains two red, two blue, two white, and two yellow balls. Susan draws balls at random from the urn without replacing them. What is the expected number of draws Susan makes until drawing her first red ball?

(A) $\dfrac{42}{14}$ **(B)** $\dfrac{43}{14}$ **(C)** $\dfrac{42}{13}$ **(D)** $\dfrac{43}{13}$ **(E)** $\dfrac{44}{13}$



11. In trapezoid $ABCD$, the area of region I is 9 and the area of region II is 16. What is the area of region III?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 12.5 **(E)** cannot be determined

12. The polynomial

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

can be factored as $P_2(x) \cdot P_6(x)$ where P_2 and P_6 are polynomials with integer coefficients and have degrees 2 and 6 respectively. Find the sum of the coefficients of P_2.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 9

13. The number of ordered pairs of integers that satisfy the equation $x^2 + 4x + y^2 = 9$ is

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

14. Let a and b be positive constants. If x is a solution of

$$\sqrt{x+a} + \sqrt{x+b} = \sqrt{a+b}$$

then x must equal

(A) 0 **(B)** $\frac{a+b}{2}$ **(C)** $-\frac{ab}{a+b}$ **(D)** $-\left(\frac{1}{a} + \frac{1}{b}\right)$ **(E)** $-\frac{2}{a+b}$

15. Which of the following lines is an asymptote of the curve $x^2 - 4xy + 3y^2 = 7$?

(A) $x + 3y = 0$ **(B)** $x - 3y = 0$ **(C)** $x + y = 0$ **(D)** $x + y = -3$ **(E)** $x - y = 1$

16. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

(A) 32 **(B)** 28 **(C)** 24 **(D)** 20 **(E)** 16

17. Evaluate the following limit:

$$\lim_{x\to\infty} \left(x^3 + ax^2\right)^{1/3} - \left(x^3 - ax^2\right)^{1/3}$$

(A) 0 **(B)** 1 **(C)** $\frac{a}{3}$ **(D)** $\frac{2a}{3}$ **(E)** a

18. In the piecewise function $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{1}{1+x^2} & \text{if } x \leq 1 \\ a(x-1)^2 + b(x-1) + c & \text{if } x > 1 \end{cases}$$

the constants a, b, and c are chosen to ensure that f is twice differentiable over the real numbers. What then must be the value of a?

(A) -2 **(B)** -1 **(C)** $-\frac{1}{2}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

19. Let $f(x)$ be differentiable with $f' > 0$. Let $g = f^{-1}$. Find the value of $(g \circ g)'(3)$ if $f(4) = 3$, $f'(4) = 2$, $f(5) = 4$, and $f'(5) = 5$.

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** 2 **(D)** 5 **(E)** 20

20. Let g be 10 times differentiable with $g(0) = 1/8!$. Suppose $f(x) = x^{10}g(x)$. Find $f^{(10)}(0)$.

(A) $\frac{1}{8!}$ **(B)** 1 **(C)** 90 **(D)** 120 **(E)** 8!