Fall 2011 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

All variables are assumed to represent real numbers unless stated in the problem otherwise.

1. The number of even factors of 7^7 is

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2. Recall that $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x^2] - [x]^2$, find $f(\pi)$.

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

3. How many subsets of $\{a, b, c, d, e\}$ have an odd number of elements?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

4. How many solutions are there to the equation $2^a = a^2$?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

5. Given that the piecewise function

$$f(x) = \begin{cases} 4x & \text{if } x \leq 0 \\ ax^2 + bx + c & \text{if } 0 < x < 1 \\ 6 - 3x & \text{if } x \geq 1 \end{cases}$$

   is differentiable at all real numbers, find the value of $a + b + c$.

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

6. The value of $13\sin(\tan^{-1}(5/12)) + 15\sin(\tan^{-1}(9/12))$ is

   (A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

7. Let $f(x)$ be differentiable at $x = 3$. If $f(3) = 5$ and $\left(\frac{1}{f}\right)'(3) = 4$ then what is the value of $f'(3)$?

(A) -100 **(B)** -20 **(C)** $-1/4$ **(D)** $1/4$ **(E)** 20

8. Determine the value of

$$\lim_{x \to 0} (\cos x)^{\frac{1}{\sin^2 x}}$$

(A) 0 **(B)** 1 **(C)** e **(D)** e^2 **(E)** $e^{-\frac{1}{2}}$

9. The graph of $y = \sin ax + \sin bx$ is shown below for x in the interval $[0, 2\pi]$. Given that a and b are positive integers, with $a + b$ large compared to $a - b$ and $a > b$, the value of $a - b$ could be:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

10. An eight by eight matrix has its (i,j)th entry given by $f(i)g(j)$ where $f(i) = 4i - 3$ and $g(j) = 2j + 5$. What is the sum of all of the entries of this matrix?

(A) 13440 **(B)** 13540 **(C)** 13640 **(D)** 13740 **(E)** 13840

11. If $f(x) = x^7 + x$, then what is the value of the second derivative of the inverse function of f at 2? That is, what is $(f^{-1})''(2)$?

(A) -7/128 **(B)** -7/16 **(C)** -21/256 **(D)** -3/64 **(E)** 3/64

12. Find the absolute maximum value of $f(x) = \sin^8 x \cos^4 x$ on the interval $[0, 2\pi]$.

(A) $\frac{2^4}{3^6}$ **(B)** $\frac{1}{2^6}$ **(C)** $\frac{3^4}{2^{12}}$ **(D)** $\frac{3^6}{2^{10}}$ **(E)** $\frac{3^2}{2^6}$

13. Five horses are in a race. In how many ways can they finish if ties are allowed?

(A) 511 **(B)** 530 **(C)** 531 **(D)** 541 **(E)** 625

14. For the sequence given by $t_{n+1} = \frac{t_n + t_{n-1} + 1}{t_{n-2}}$, with $t_1 = 4$, $t_2 = 2$ and $t_3 = 5$, find t_{2011}.

(A) 4 **(B)** 2 **(C)** 5 **(D)** 7/5 **(E)** 16/5

15. Find the radius of the sphere that contains the points $(0, -3, -2)$, $(0, -3, 2)$, $(2, 3, 1)$, and $(-2, 3, 1)$.

(A) $\frac{43}{12}$ **(B)** $\frac{\sqrt{1945}}{12}$ **(C)** $\frac{47}{12}$ **(D)** $\frac{\sqrt{2011}}{12}$ **(E)** $\frac{49}{12}$

16. A certain polynomial $P(x)$ has the property that all its coefficients are non-negative integers, none of which is larger than 6. If $P(7) = 2011$, what is P's coefficient of x^2?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

17. If $a = \dfrac{1110}{1111}$, $b = \dfrac{2221}{2223}$, and $c = \dfrac{3331}{3334}$ which of the following is true?

(A) $a > b > c$ **(B)** $b > a > c$ **(C)** $c > a > b$ **(D)** $c > b > a$ **(E)** $b > c > a$

18. Which of the following are always true about the pair of functions $y = a^x$ and $y = \log_a x$, where $a > 1$?

I. Both are increasing on their domains

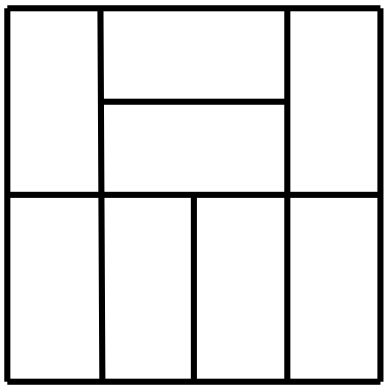
II. For the same $a > 1$, their graphs never intersect

III. For the same $a > 1$, they are inverses of each other.

(A) I only **(B)** II only **(C)** I and II only
(D) I and III only **(E)** I, II and III

19. In how many ways can a 4×4 nailed down board be tiled by eight 1×2 dominoes? One way to tile the board is shown below.

(A) 16 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 49



20. If $a + b + c = 9$ and $ab + bc + ca = 7$ then the maximum possible value of c is closest to

(A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10