Spring 2012 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. The value of $-3-3^2-3^3$ is equal to

   (A) -41 **(B)** -40 **(C)** -39 **(D)** -37 **(E)** -36

2. What is the largest possible product of two positive odd integers whose sum is 40?

   (A) 39 **(B)** 279 **(C)** 300 **(D)** 399 **(E)** 400

3. If $a \diamond b$ equals the lesser of $1/a$ and $1/b$, find the value of $-3 \diamond (-2 \diamond (-1/2))$.

   (A) -3 **(B)** -1/3 **(C)** -1/2 **(D)** -2 **(E)** -1

4. What is the remainder when the sum

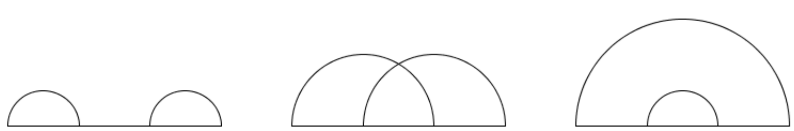
$$1^{111}+2^{111}+3^{111}+4^{111}+5^{111}+6^{111}+7^{111}+8^{111}+9^{111}+10^{111}$$

   is divided by 11?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

5. Distribute 14 points along a line segment. How many distinct ways exist for pairing the points via semicircles? The case of four points is pictured below.

   (A) 10395 **(B)** 40320 **(C)** 60125 **(D)** 101245 **(E)** 135135



6. If $2 + \ln x = \ln(x+2)$ then x must equal

(A) $\dfrac{2}{e^2-1}$ **(B)** $\dfrac{2}{e-1}$ **(C)** $\dfrac{1}{e^2-1}$ **(D)** $\dfrac{2}{e^2+1}$ **(E)** $\dfrac{1}{e^2+1}$

7. When expanded and simplified $(1 - x + x^2 - x^3)^{10}$ has the form

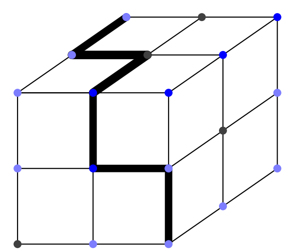
$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{30} x^{30}$$

What is the value of $c_1 + c_3 + c_5 + \cdots + c_{29}$, the sum of the coefficients of all the odd powers of x?

(A) -4^{19} **(B)** -2^{20} **(C)** -2^{19} **(D)** 0 **(E)** 2^{19}

8. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

(A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120



9. Find the value of $r^2s^2 + s^2t^2 + t^2r^2$ if r, s, and t are the three possibly complex roots of the cubic polynomial $x^3 + 5x^2 - 3x + 1$.

(A) -1 **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 8

10. What is the sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \cdots$$

which consists of the sum of the reciprocals of 1 and all the natural numbers whose prime factorizations contain no primes other than 2 or 5?

(A) $5/2$ **(B)** $7/2$ **(C)** $5/3$ **(D)** 2 **(E)** 3

11. In $\triangle ABC$, the medians AD and BE are perpendicular. If $AC = 8$ and $BC = 12$, what is the length of AB?

(A) 6 **(B)** 9 **(C)** $4\sqrt{3}$ **(D)** 7 **(E)** $4\sqrt{13/5}$

12. When the polynomial x^{2012} is divided by the polynomial $x^2 + x + 1$ what is the remainder $R(x)$?

(A) 1 **(B)** $x + 1$ **(C)** $2x - 1$ **(D)** 0 **(E)** $-x - 1$

13. Which of the following are equal to $\int_a^b f(x)\,dx$ for all continuous functions f and all values of the constants a, b, and k, with $k \neq 0$?

I. $-\int_b^a f(x)\,dx$

II. $\int_a^b f(a + b - x)\,dx$

III. $\int_{ka}^{kb} f(kx)\,dx$.

(A) none of them **(B)** I only **(C)** I and II only
(D) I and III only **(E)** I, II and III

14. When the integrals listed below are arranged in order from least to greatest, which integral will be in the center?

$$\int_0^1 1 + x\,dx \qquad \int_0^1 \frac{1}{1+x}\,dx \qquad \int_0^1 \frac{1}{1 + x + \frac{x^2}{2}}\,dx \qquad \int_0^1 e^x\,dx \qquad \int_0^1 e^{-x}\,dx$$

(A) $\int_0^1 1 + x\,dx$ **(B)** $\int_0^1 \frac{1}{1+x}\,dx$ **(C)** $\int_0^1 \frac{1}{1 + x + \frac{x^2}{2}}\,dx$
(D) $\int_0^1 e^x\,dx$ **(E)** $\int_0^1 e^{-x}\,dx$

15. If $f(x)$ and $g(x)$ are differentiable functions and $F(x) = \int_0^{g(x)} f(t)\,dt$, then $F'(1)$ equals

(A) $f(g(1))$ **(B)** $f(1)g'(1)$ **(C)** $f'(g(1))g'(1)$ **(D)** $f(g(1))g'(1)$ **(E)** $f'(1)g'(1)$

16. Evaluate $\int_0^{49} \frac{1}{\sqrt{16-\sqrt{x}}}\,dx$.

(A) 28/3 **(B)** 32/3 **(C)** 12 **(D)** 40/3 **(E)** 44/3

17. Given that the area of an ellipse with semimajor axis a and semiminor axis b is πab, find the volume of the set of points $\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \leq 1\}$.

(A) 10π **(B)** 20π **(C)** 30π **(D)** 40π **(E)** 50π

18. John has just learned the arclength formula for functions of the form $y = f(x)$ and wishes to test this formula by measuring with a string the actual graph of $y = x^2$ from $x = 0$ to $x = 1$, with one unit in both the x and y direction measuring one inch. However, when he prints the graph of $y = x^2$ using his Computer Algebra System, he notes that while the unit in the x direction does measure exactly one inch, the unit in the y direction measures only three-quarters of an inch. Which integral below will give the length in inches of the actual printed curve $y = x^2$ from $x = 0$ to $x = 1$?

(A) $\int_0^1 \sqrt{1 + \frac{9}{4}x^2}\,dx$ **(B)** $\int_0^1 \sqrt{1 + \frac{4}{9}x^2}\,dx$ **(C)** $\int_0^1 \sqrt{1 + 4x^2}\,dx$
(D) $\int_0^1 \sqrt{1 + 9x^2}\,dx$ **(E)** $\int_0^1 \sqrt{1 + 6x^2}\,dx$

19. Given that $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots$ and $\ln k = 1 + 1/2 + 1/3 - 3/4 + 1/5 + 1/6 + 1/7 - 3/8 + \cdots + 1/(4n+1) + 1/(4n+2) + 1/(4n+3) - 3/(4n+4) + \cdots$, then $k =$

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

20. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

(A) 5/12 **(B)** 1/11 **(C)** 1/24 **(D)** 1/22 **(E)** 1/48