Fall 2013 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed

1. How many positive factors does 24 have?

2. How many real numbers are equal to their own reciprocal?

3. The *Fibonacci* sequence is
$$1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$
where each term past the second equals the sum of the two previous terms. Write down the first perfect square that appears in the Fibonacci sequence after the first two 1's.

4. Calculate $(17 + 62 + 83 + 38)^2$.

5. Hezy is a cashier and has more than 50 pennies and more than ten dimes in his cash register but only one quarter and no nickels or half-dollars. In how many different ways can he hand you 50 cents in change?

6. Find the smallest integer greater than $\sqrt[4]{4217}$.

7. Two positive integers have a product of 210 and a sum of 29. What is the result when the larger of the two integers is subtracted from the smaller of the two integers?

8. How many zeroes does $\dfrac{53!}{39!}$ end in?

9. What is the 2013th digit in the decimal expansion of $3/7$?

10. A satellite passes over Dallas at 3:00pm on a Tuesday. If the satellite orbits the earth every 11 hours, what is the day of the week the very next time it passes over Dallas at 3:00pm?

11. How many six-digit integers of the form $79A4B1$, where A and B are digits, are divisible by eleven?

12. What is the largest prime p satisfying $\left(\dfrac{16}{p}\right)^2 > 2$?

13. With a, b, and c positive integers let $\text{lcm}(a, b, c)$ stand for the least common multiple of a, b, and c. If $4{\cdot}\text{lcm}(ab, bc, ca) = abc$, what is least possible value of $\text{lcm}(a, b, c)$?

14. Which of these numbers is the largest?
$$2^{40} \qquad 3^{25} \qquad 5^{18} \qquad 7^{13}$$

15. When listing the permutations of the string of letters $ABCD$ we decide to put them in alphabetical order. So, for example, the string $CBAD$ will come after the string $CADB$. In which place does the string $CDAB$ occur? Answer like '45th'.

Fall 2013 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. How many real numbers are equal to their own reciprocal?
2. Two positive integers have a product of 210 and a sum of 29. What is the result when the larger of the two integers is subtracted from the smaller of the two integers?
3. What is the largest prime p satisfying $\left(\frac{16}{p}\right)^2 > 2$?
4. A certain parking lot charges 5 dollars for the first hour or portion of, and 2 dollars per following hour or portion of. If Susan parks her car at 7:41am and drives out at 2:59pm the same day, how much does she owe?
5. In a square pyramid, the sides of the square base are doubled and the height is halved. By what percent is the volume of the original pyramid changed?
6. If one paper clip costs p cents and three erasers costs q cents, how many cents do 21 erasers and 12 paper clips cost? Answer in terms of p and q.
7. Ashley's creature box for her science experiment contains centipedes and spiders. Despite the name 'centipede'each of Ashley's centipedes has 30 legs. She counts a total of 23 insects and 228 legs. How many centipedes does Ashley have? By the way, spiders have 8 legs!
8. A *unit fraction* is a fraction of the form $1/n$ where n is a positive integer. Write $3/17$ as a sum of unit fractions, each with a different denominator.
9. Find the coefficient of x^2 when $(1 + x + x^2)^6$ is expanded and simplified.
10. If x ounces of cleaner clean y square feet of floor, how many square yards of floor can be cleaned by y gallons of this cleaner?
11. Find the value of x if $4^5 + 4^5 + 4^5 = 2^x + 2^x + 2^x$.
12. For which integer m does $\frac{m}{13} < \sqrt{2} < \frac{m+1}{13}$ hold?
13. Simplify
$$a - \Big(3b - \big(a - (2b - a)\big)\Big)$$
14. A set S of ordered pairs is said to be *transitive* if whenever (a, b) and (b, c) belong to S then so does (a, c). Is this set S below transitive?
$$S = \{(6,13), (4,8), (5,7), (6,10), (3,5), (10,13), (3,7), (1,5), (3,10), (1,4), (1,7), (9,6)\}$$
Answer Yes or No.
15. Find the maximum possible product of a set of positive integers whose sum is 27. Answer in standard integer form.

Fall 2013 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. In a square pyramid, the sides of the square base are doubled and the height is halved. By what percent is the volume of the original pyramid changed?

2. Ashley's creature box for her science experiment contains centipedes and spiders. Despite the name 'centipede'each of Ashley's centipedes has 30 legs. She counts a total of 23 insects and 228 legs. How many centipedes does Ashley have? By the way, spiders have 8 legs!

3. Given that $b > a$, write down the least of these numbers:
$$\frac{4b+a}{5}, \qquad \frac{3a+7b}{10}, \qquad \frac{a+8b}{9}, \qquad \frac{4b+3a}{7}, \qquad \frac{5a+5b}{10}$$

4. The sum of the supplements of the angles of a triangle always equals what? Answer in degrees.

5. Find the area of the region $T = \{(x, y) : |x| + 3|y| \leq 4\}$.

6. Find the integer m such that
$$(\sqrt{2}-1)^4 = \sqrt{m} - \sqrt{m-1}$$

7. Segment AD is an altitude of equilateral triangle ABC and segment DE is an altitude of triangle CDA. Find the ratio AE/EC.

8. Determine how many ordered pairs of positive integers (a, b) satisfy $\frac{1}{a} + \frac{2}{b} = \frac{3}{7}$.

9. In quadrilateral $ABCD$, $\angle BAC = 30^\circ$, $\angle CAD = 50^\circ$, and $BA = CA = DA$. Find the measurement of $\angle BDA$ in degrees.

10. A point (a, b) is first reflected across the x-axis, then across the line $y = -x$ and finally across the origin to land at the point $(12, 5)$. Find the sum $a + b$.

11. In how many different ways can 10 identical chairs be distributed to 4 distinct tables? A table may be left without any chairs at all. Answer in standard integer form.

12. In $\triangle ABC$ we have $AB = 13$, $AC = 15$, and $BC = 14$. Find the length of the median from vertex A to side BC.

13. If the point (x, y) satisfies
$$x^3 - 71x = y^3 - 71y$$
but does not satifsy $x = y$ then what is the value of $x^2 + xy + y^2$?

14. Point A has coordinates $(19, -104/3)$ while point B has coordinates $(-43/3, -14)$. Find the coordinates of a point C which lies on the perpendicular bisector of segment AB given that both of the coordinates of C are integers.

15. Let $f(x, y) = yx^2 - (2y+1)x + y$. Solve $f(x, 6) = 0$.

Fall 2013 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. If x ounces of cleaner clean y square feet of floor, how many square yards of floor can be cleaned by y gallons of this cleaner?

2. A set S of ordered pairs is said to be *transitive* if whenever (a, b) and (b, c) belong to S then so does (a, c). Is this set S below transitive?

$$S = \{(6,13), (4,8), (5,7), (6,10), (3,5), (10,13), (3,7), (1,5), (3,10), (1,4), (1,7), (9,6)\}$$

   Answer Yes or No.

3. Find the coefficient of x^2 when $(1 + x + x^2)^6$ is expanded and simplified.

4. Find the area of the region $T = \{(x, y) : |x| + 3|y| \leq 4\}$.

5. Two congruent circles (in the same plane) do not intersect. Their centers are a distance 10 units apart. The length of their common internal tangent is 8 units. What is the radius of this pair of congruent circles?

6. Segment AD is an altitude of equilateral triangle ABC and segment DE is an altitude of triangle CDA. Find the ratio AE/EC.

7. Let $f(x, y) = yx^2 - (2y + 1)x + y$. Solve $f(x, 6) = 0$.

8. If the point (x, y) satisfies
$$x^3 - 71x = y^3 - 71y$$
   but does not satifsy $x = y$ then what is the value of $x^2 + xy + y^2$?

9. Find the set of all values of the parameter a so that the graph of the parabola $y = ax^2 + 2x + 4a$ never enters the third quadrant III. Recall that $III = \{(x, y) : x < 0 \text{ and } y < 0\}$.

10. Find the minimum possible value of the expression
$$(x-9)^2 + (x-7)^2 + (x-5)^2 + (x+5)^2 + (x+7)^2 + (x+9)^2$$

11. Find the value of the index n if
$$\sqrt[3]{r \sqrt[n]{\left(\frac{1}{r}\right) \sqrt[4]{r}}} = r^{\frac{7}{24}}$$

12. In $\triangle ABC$ with point D on segment AC so that $AD : DC = 2 : 5$, draw segment BD with points E and F on BD in order $B - E - F - D$. If the area of $\triangle EFC$ is 5 and the area of $\triangle ABC$ is 30, find the area of $\triangle AEF$.

13. Let m and n be two relatively prime positive integers. Find the maximum possible value of the gcd of $m + 24n$ and $n + 24m$.

14. In which quadrant (I, II, III, or IV) of the complex plane does $(-\sqrt{3} + 3i)^{17}$ lie?

15. The point $(5, -4)$ lies on the graph of $y = -2f(3(x + 6)) - 8$. What corresponding point lies on the graph of $y = (1/2)f(x - 2) + 5$?

Fall 2013 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. How many six-digit integers of the form $79A4B1$, where A and B are digits, are divisible by eleven?

2. A set S of ordered pairs is said to be *transitive* if whenever (a,b) and (b,c) belong to S then so does (a,c). Is this set S below transitive?

$$S = \{(6,13),(4,8),(5,7),(6,10),(3,5),(10,13),(3,7),(1,5),\\ (3,10),(1,4),(1,7),(9,6)\}$$

   Answer Yes or No.

3. Find the value of x if $4^5+4^5+4^5 = 2^x+2^x+2^x$.

4. For which integer m does $\dfrac{m}{13} < \sqrt{2} < \dfrac{m+1}{13}$ hold?

5. In how many different ways can 10 identical chairs be distributed to 4 distinct tables? A table may be left without any chairs at all. Answer in standard integer form.

6. Determine how many ordered pairs of positive integers (a,b) satisfy $\dfrac{1}{a}+\dfrac{2}{b}=\dfrac{3}{7}$.

7. Find the set of all values of the parameter a so that the graph of the parabola $y = ax^2+2x+4a$ never enters the third quadrant III. Recall that $III = \{(x,y) : x < 0 \text{ and } y < 0\}$.

8. Point A has coordinates $(19,-104/3)$ while point B has coordinates $(-43/3,-14)$. Find the coordinates of a point C which lies on the perpendicular bisector of segment AB given that both of the coordinates of C are integers.

9. Find the largest value of c for which the following inequality holds for all x:

$$x^4+7x^2+144 \geq cx^2$$

10. Find the value of n if $\log_{48} 12 = \log_n 144$.

11. Find all real solutions of the equation

$$0 = (2x+1)^5+(2x+1)^4(2x-1)+(2x+1)^3(2x-1)^2\\ +(2x+1)^2(2x-1)^3+(2x+1)(2x-1)^4+(2x-1)^5$$

12. In a triangle one of the angles is twice another and the sides opposite these angles have lengths 16 and 28. Find the length of the third side of this triangle.

13. Find the number of ordered pairs of integers (m,n) that satisfy

$$m^2-mn+n^2-m-n=0$$

14. Find the maximum possible product of a set of positive integers whose sum is 27. Answer in standard integer form.

15. Let z be a complex number. If $z=a+bi$ where a and b are real numbers, find how many such z with $b<0$ satisfy

$$z^7+z^5+z^3+z=9$$

Spring 2013 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. What is the largest prime p satisfying $\left(\dfrac{16}{p}\right)^2 > 2$?

2. Find the value of x if $4^5 + 4^5 + 4^5 = 2^x + 2^x + 2^x$.

3. Find the integer m such that
$$(\sqrt{2} - 1)^4 = \sqrt{m} - \sqrt{m-1}$$

4. Find the value of n if $\log_{48} 12 = \log_n 144$.

5. In how many different ways can 10 identical chairs be distributed to 4 distinct tables? A table may be left without any chairs at all. Answer in standard integer form.

6. In which quadrant (I, II, III, or IV) of the complex plane does $(-\sqrt{3} + 3i)^{17}$ lie?

7. Suppose that for all $1 < x < 3$
$$|7x + 4 - f(x)| \leq (|x-2|)^{3/2}$$
Find the value of $f'(2)$.

8. Find all values of the parameter r so that $y(x) = e^{rx}$ satisfies $y''(x) + 5y'(x) = 14y(x)$ for all x.

9. Let $f(x) = x^4 + ax^3 + (a+2)x^2$. On what maximal open interval or union of open intervals of the parameter a is $f(x)$ concave up for all x.?

10. If f is a differentiable function on $[-3, 3]$ satisfying $f(1) = -1$, $f(-1) = 1$, $f'(1) = 2$ and $f'(-1) = -2$, find the value of $(f \circ f \circ f \circ f)'(1)$.

11. Find the largest value of the constant c so that
$$x^4 + 9 \geq cx$$
holds for all x.

12. Let g be twice-differentiable on the interval $[0, 5]$. Let $g(0) = g'(0) = 0$ and suppose $g''(x) \leq 6$ for all x in $[0, 5]$. What is the maximum possible value of $g(4)$?

13. The set of all tangent lines to a function $f(x)$ can be described as
$$\{y = a(2x - a) + 2 - 4x \,:\, a \in \mathbb{R}\}$$
Find an algebraic formula for this function $f(x)$ in terms of x.

14. Find $g'''(0)$ if
$$g(x) = \frac{x+1}{1 - x^3 - x^4}$$

15. Determine the value of
$$\lim_{x \to 0} \frac{\ln(1 + 2x + x^2) + \ln(1 - 2x + x^2)}{x(e^x - 1)}$$