Spring 2019 McNabb GDCTM Contest
Pre-Algebra

No calculators are allowed. You have 45 minutes. Enjoy the problems!

1. Simplify
$$200\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}\right)$$

2. A store raised the price of a dozen eggs from \$2.50 to \$2.85. What was the percent increase?

3. How many numbers are in the list
$$-21, -17, -13, \cdots, 499$$
?

4. In the state of Xlandis the first 10,000 dollars of an inheritance is not taxed. After that, the state takes one-third of the remaining inheritance. If someone wishes to make sure their heir actually gets 20,000 dollars, how much should be left to that heir in the will?

5. Write 52 as the sum of three distinct primes. Only one way of doing this needs to be written down.

6. Sam bought 5 pens and 4 notebooks at the store. If each pen cost 84 cents and each notebook cost \$3.17, how much change did Sam get if he paid with a twenty dollar bill?

7. If Hezy drives for three hours at 30 miles per hour followed by five hours at 46 miles per hour, what is his average speed in miles per hour for the whole trip?

8. In many ways can the letters in the word MACAROON be arranged?

9. Solve the equation
$$x+\frac{x}{3}+\frac{x}{5}=0$$

10. Simplify $\dfrac{30!-29!}{31!-30!}$.

11. What is the number of positive factors of the number 20^{19}?

12. A Math Club has eleven eighth-graders, eight seventh-graders, and seven sixth-graders. In how many ways can this club put together a team that has two students from each of these grades?

13. Sandra reads a ten-volume history of Texas. Volume One is 110 pages long. Each subsequent volume is ten pages longer than the previous volume. She reads 50 pages per day except when reaching the end of a volume she stops reading for that day. How many days does it take Sandra to finish reading all ten volumes of this history of Texas?

14. What is the number of subsets of $\{1, 2, 3, 4, \cdots, 99, 100\}$ with an even number of even numbers? Recall that 0 is an even number. Answer in the form 2^n where n is an integer.

15. Find the number of ways to color the edges of a square if four colors are available and two colorings are considered the same if one can be rotated into the other.

Spring 2019 McNabb GDCTM Contest
Algebra One

No calculators are allowed. You have 45 minutes. Enjoy the problems!

1. Find the prime factorization of $8^{11}9^{15}$.

2. In many ways can the letters in the word MACAROON be arranged?

3. Solve the equation
$$x + \frac{x}{3} + \frac{x}{5} = 0$$

4. Sam bought 5 pens and 4 notebooks at the store. If each pen cost 84 cents and each notebook cost \$3.17, how much change did Sam get if he paid with a twenty dollar bill?

5. Find one ordered pair of positive integers (m, n) which satisfies the equation $m^2 - 5n^2 = 1$.

6. If Hezy drives for three hours at 30 miles per hour followed by five hours at 46 miles per hour, what is his average speed in miles per hour for the whole trip?

7. Sandra reads a ten-volume history of Texas. Volume One is 110 pages long. Each subsequent volume is ten pages longer than the previous volume. She reads 50 pages per day except when reaching the end of a volume she stops reading for that day. How many days does it take Sandra to finish reading all ten volumes of this history of Texas?

8. The sum of two integers is 61. What is the largest possible value of their product?

9. A group of students plan to contribute equally to a charity. They have decided in advance the total amount of their contribution. If three students drop out of this plan, the remaining students would have to each contribute two dollars more to preserve the original gift. If six of the original students were to drop out of this plan, the remaining students would each have to contribute five dollars more to preserve the original gift. How many students are in the original group?

10. Simplify
$$(x + y - z)(x + y + z)$$

11. The graph of the equation $9x^2 - 16y^2 = 0$ is the union of two lines. What is the product of the slopes of those lines?

12. Four standard cubical fair dice are rolled. The probability of getting two distinct pairs (such as two 3's and two 5's) can be expressed in the form m/n, where m and n are relatively prime positive integers. Find the value of $m + n$.

13. What is the number of subsets of $\{1, 2, 3, 4, \cdots, 99, 100\}$ with an even number of even numbers? Recall that 0 is an even number. Answer in the form 2^n where n is an integer.

14. Find the number of ways to color the edges of a square if four colors are available and two colorings are considered the same if one can be rotated into the other.

15. Let a and b solve the system

$$\begin{cases} a + \dfrac{1}{b} & = 7 \\ b + \dfrac{1}{a} & = 9/14 \end{cases}$$

Find the sum of all possible values of ab.

Spring 2019 McNabb GDCTM Contest
Geometry

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. Solve the equation
$$x + \frac{x}{3} + \frac{x}{5} = 0$$

2. If Hezy drives for three hours at 30 miles per hour followed by five hours at 46 miles per hour, what is his average speed in miles per hour for the whole trip?

3. Simplify $\dfrac{30! - 29!}{31! - 30!}$.

4. A piece of wire is wrapped in a spiral around a cylinder of radius $3/\pi$ and height 12 in such a way that it completes two revolutions as it goes from bottom to top of the cylinder. How long is the wire?

5. The angle bisectors of the angles of $\triangle ABC$ meet at point D. If $\angle A = 100°$, find the measure of $\angle BDC$ in degrees.

6. A group of students plan to contribute equally to a charity. They have decided in advance the total amount of their contribution. If three students drop out of this plan, the remaining students would have to each contribute two dollars more to preserve the original gift. If six of the original students were to drop out of this plan, the remaining students would each have to contribute five dollars more to preserve the original gift. How many students are in the original group?

7. Find the number of ways to color the edges of a square if four colors are available and two colorings are considered the same if one can be rotated into the other.

8. In $\triangle ABC$ points D and E are on sides $\overline{BC}$ and $\overline{AC}$ respectively. Draw segments $\overline{AD}$ and $\overline{BE}$ intersecting at point F. If $AE/EC = 3$ and $BF/FE = 8/3$, then find the value of AF/FD.

9. What is the number of subsets of $\{1, 2, 3, 4, \cdots, 99, 100\}$ with an even number of even numbers? Recall that 0 is an even number. Answer in the form 2^n where n is an integer.

10. Eight spheres, each of radius one, are situated such that their centers form the vertices of a cube and each sphere is externally tangent to exactly three of the other spheres. A ninth sphere is externally tangent to all of the original eight spheres. Find the radius of this ninth sphere.

11. Reflection across the line $y = x/\sqrt{3}$ followed by reflection across the line $y = x\sqrt{3}$ is equivalent to a counter-clockwise rotation of θ degrees, where $0 < \theta < 180$. Find the value of θ.

12. How many paths are there from the point $(0, 0, 0, 0)$ to the point $(2, 2, 2, 2)$ if the only possible moves are to increase just a single coordinate by 1?

13. Two medians of a triangle both have length 9. If the area of the triangle is $24\sqrt{5}$, find the largest possible length of its third median.

14. Given $\triangle ABC$ with area 13, points D, E, and F are located on sides BC, CA, and AB respectively in such a way that $BD/DC = CE/EA = AF/FB = 3/1$. Segments AD, BE, and CF are drawn intersecting each other pairwise in three points G, H, and I. Find the area of $\triangle GHI$.

15. Define the sum of two points $P(a, b)$ and $Q(c, d)$ in the plane to be as if they were vectors, namely, $P + Q = (a + c, b + d)$. Let S and T be two regions in the plane. Define their sum as

$$S + T = \{P + Q \mid P \in S,\ Q \in T\}$$

Let

$$S = \{(x, y) \mid x^2 + y^2 \leq 1\}$$
$$T = \{(x, y) \mid 4 \leq x \leq 5 \quad \text{and} \quad 0 \leq y \leq 1\}$$

Find the area of $S + T$.

Spring 2019 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. Write 52 as the sum of three distinct primes. Only one way of doing this needs to be written down.

2. How many numbers are in the list
$$-21, -17, -13, \cdots, 499$$
?

3. Sam bought 5 pens and 4 notebooks at the store. If each pen cost 84 cents and each notebook cost \$3.17, how much change did Sam get if he paid with a twenty dollar bill?

4. A group of students plan to contribute equally to a charity. They have decided in advance the total amount of their contribution. If three students drop out of this plan, the remaining students would have to each contribute two dollars more to preserve the original gift. If six of the original students were to drop out of this plan, the remaining students would each have to contribute five dollars more to preserve the original gift. How many students are in the original group?

5. What is the range of the function $f(x) = \ln(1 + 2^x)$? Answer in interval notation.

6. The angle bisectors of the angles of $\triangle ABC$ meet at point D. If $\angle A = 100°$, find the measure of $\angle BDC$ in degrees.

7. What is the period of the trigonometric function
$$f(x) = \sin 5x + \cos 4x + \tan 3x$$
? Note that x is assumed to be in radians.

8. Let a and b solve the system
$$\begin{cases} a + \dfrac{1}{b} & = 7 \\ b + \dfrac{1}{a} & = 9/14 \end{cases}$$
Find the sum of all possible values of ab.

9. A piece of wire is wrapped in a spiral around a cylinder of radius $3/\pi$ and height 12 in such a way that it completes two revolutions as it goes from bottom to top of the cylinder. How long is the wire?

10. The sides of $\triangle ABC$ have lengths $AB = \sin 27°$, $BC = \cos 10°$, and $CA = \cos 17°$. Find the measure of $\angle A$ in degrees.

11. Solve

$$\sqrt{3+x}+\sqrt{3-x}>3$$

Answer in interval notation.

12. When the binomial $\left(2a+\dfrac{1}{\sqrt{a}}\right)^9$ is expanded and simplified, what is the value of the constant term?

13. Two ladders, one of length 25 feet the other of length 17 feet criss-cross each other across an alley. The foot of each ladder is up against the base of the walls facing each other. The point at which the two ladders cross each other is 40/7 feet above the alley. How wide is the alley?

14. Let z be a complex number. Find the solutions of

$$z^2-(3-i)z+(8+i)=0$$

Note that here $i=\sqrt{-1}$. Answers must be in standard complex form $a+bi$ where a and b are real numbers.

15. Solve

$$6(\log_8 x)\cdot(\log_2 x)+6\log_4 x=-1$$

Spring 2019 McNabb GDCTM Contest
Pre-Calculus

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. A store raised the price of a dozen eggs from \$2.50 to \$2.85. What was the percent increase?

2. Find one ordered pair of positive integers (m, n) which satisfies the equation $m^2 - 5n^2 = 1$.

3. Let $i = \sqrt{-1}$. Simplify $(1+i)^3$.

4. Let $\sin\theta = 8/17$ and $\cos\theta = 15/17$. Find the value of $\tan 2\theta$.

5. Let $p(x) = x^4 - 2x^3 + 3x^2 - 2x + 5$. Find the sum of the coefficients of $p(2x)$.

6. Let

$$f(x) = \begin{cases} 2x+4 & \text{if } x \le -2 \\ \dfrac{1}{2}x + 1 & \text{if } x > 2 \end{cases}$$

   Find the value of $f^{-1}(7)$.

7. Define the sum of two points $P(a, b)$ and $Q(c, d)$ in the plane to be as if they were vectors, namely, $P + Q = (a + c, b + d)$. Let S and T be two regions in the plane. Define their sum as

$$S + T = \{P + Q \mid P \in S,\ Q \in T\}$$

   Let

$$S = \{(x, y) \mid x^2 + y^2 \le 1\}$$
$$T = \{(x, y) \mid 4 \le x \le 5 \quad \text{and} \quad 0 \le y \le 1\}$$

   Find the area of $S + T$.

8. Simplify

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^8$$

9. Consider an infinite geometric series consisting of positive terms such that the sum of the first three terms is 39 and the sum of the reciprocals of the first three terms is 13/27. Find the sum of this infinite series.

10. Find the number of ways to color the edges of a square if four colors are available and two colorings are considered the same if one can be rotated into the other.

11. How many paths are there from the point $(0, 0, 0, 0)$ to the point $(2, 2, 2, 2)$ if the only possible moves are to increase just a single coordinate by 1?

12. Solve

$$6(\log_8 x) \cdot (\log_2 x) + 6\log_4 x = -1$$

13. Let x be an angle measured in radians such that $0 < x < \pi/2$ and

$$\cos x = \frac{1}{2} \cdot \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

Find the value of x.

14. Let z stand for a complex number. Find the area of the region in the complex plane which consists of all z such that

$$|z - 2|^2 + |z + 2|^2 \leq 26$$

15. Completely factor the polynomial

$$a^2(b + c) + b^2(c + a) + abc - c^3$$

Spring 2019 McNabb GDCTM Contest
Calculus

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. The angle bisectors of the angles of $\triangle ABC$ meet at point D. If $\angle A = 100°$, find the measure of $\angle BDC$ in degrees.

2. What is the number of positive factors of the number 20^{19}?

3. Find the value of the limit
$$\lim_{n\to\infty} \frac{3n+2}{2n+3}$$

4. If $\cos 2\theta = 1/9$, find the least possible value of $\sin\theta$.

5. Find the equation of the tangent line to $x^3 + y^3 = \dfrac{28xy}{3}$ if the point of tangency is $(3, 1)$. Answer in form $y = mx + b$.

6. For what values of the constant a does $y = \sin(ax)$ solve the differential equation $y'' + 9y = 0$?

7. Let $f(x)$ be continuous on the interval $[0, 5]$. If $\displaystyle\int_0^4 f(x)\,dx = 3$ and $\displaystyle\int_4^5 f(x)\,dx = -7$, find the value of $\displaystyle\int_0^5 f(x)\,dx$

8. Find the value of $\displaystyle\int_0^{\pi/6} \sin 2x \cos 3x\,dx$

9. Let $f(x)$ be differentiable on the interval $[0, 1]$. If $\displaystyle\int_0^1 xf(x)\,dx = 3$ and $\displaystyle\int_0^1 x^2 f'(x)\,dx = 4$, then find the value of $f(1)$.

10. Find the value of $\displaystyle\int_2^4 \frac{2x}{x^4-1}\,dx$

11. A spherical solid of radius 6 cm has density $\rho(r) = 10/r$ gm/cm^3, where r is the distance to the center of the sphere in cm. What is the mass in grams of this solid?

12. For what value of the real parameter a does the polynomial $x^4 - 3x^3 - 6x^2 + ax - 24$ have a double root?

13. Determine the values of the real-valued parameters a and b which minimize $\displaystyle\int_0^1 (x^2 - ax - b)^2\,dx$. Then put in the answer box the number $a + b$.

14. Let $F(x, y) = x^2 + y^2$ and $\Omega = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Find the average value of $F(x, y)$ over the set Ω.

15. Find all values of the real parameter a such that the cubic $x^3 - 2x^2 + x + a$ has only real roots. Answer in interval notation form.