Spring 2010 McNabb GDCTM Contest
Level I Solutions

1. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

   Solution: **(C)** Let the plums be numeraire, that is, assign plum equal to one. Let a pomegranate weigh x and a pineapple weigh y. Solve the system $3x + y = 16$, $4 + x = y$, for example, by substitution. We get $3x + 4 + x = 16$ or $x = 3$. Then $y = 7$. So three pineapples weigh 21 and it takes 7 pomegranates to weigh as much.

2. What is the area of the quadrilateral in the coordinate plane with vertices whose coordinates are (in order): $(0, 0)$, $(7, 1)$, $(4, 4)$, and $(2, 11)$?

   (A) 30 **(B)** 31 **(C)** 31.5 **(D)** 32 **(E)** 33

   Solution: **(A)** This can be done with the shoestring method. Setting up the determinants by going counterclockwise around the quadrilateral gives for the area

$$\frac{28 + 44 - 4 - 8}{2} = 60/2 = 30$$

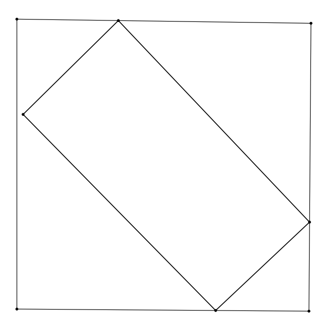
3. Ronald has an unlimited number of 5 cent and 7 cent stamps. What is the largest amount of postage (in cents) that he cannot make with these stamps?

   (A) 16 **(B)** 22 **(C)** 23 **(D)** 79 **(E)** 99

   Solution: **(C)** Notice that 24, 25, 26, and 27 cent postage can all be made by Ronald. Once 4 in a row are established like this, all higher amounts of postage is possible, simply by adding the right number of 4 cent stamps. Since 23 cents of postage cannot be made, 23 is then the largest such amount.

4. A rectangle with unequal sides is placed in a square so that each vertex of the rectangle lies on a side of the square at a trisection point of that side as shown. What is the fraction of the area of the square that is covered by the rectangle?

   (A) 1/3 **(B)** 7/18 **(C)** 4/9 **(D)** 1/2 **(E)** 5/9



   Solution: **(C)** Suppose the sides of the square have length 3. The two smaller triangles outside the rectangle can be glued together to form a square of area 1. Likewise the two larger triangles can be glued to form a square of area 4. Subtraction from the area of the large square shows the rectangle has area 4. Thus the rectangle covers 4/9 of the square.

5. The area of a triangle with sides of length 13, 14, and 15 is closest to

   (A) 84 **(B)** 86 **(C)** 88 **(D)** 90 **(E)** 92

   Solution: **(A)** Use Heron's formula. The semiperimeter s equals 21, so the area squared is $21(21-13)(21-14)(21-15) = 21 \cdot 8 \cdot 7 \cdot 6 = 7^2 \cdot 3^2 \cdot 4^2$. The area is $7 \cdot 3 \cdot 4 = 84$.

6. If $f(x)$ is a linear function for which $f(8) - f(1) = 11$, then $f(41) - f(6)$ is equal to

   (A) 61 **(B)** 55 **(C)** 49 **(D)** 43 **(E)** 37

   Solution: **(B)** From the first equation the slope of the line must be 11/7. Therefore $f(41) - f(6) = (11/7)(41-6) = (11/7)35 = 11 \cdot 5 = 55$.

7. The surface area of a large spherical balloon is doubled. By what factor is the volume of the balloon increased?

   (A) 8 **(B)** 4 **(C)** $2\sqrt{2}$ **(D)** $\sqrt[3]{4}$ **(E)** 2

   Solution: **(C)** Since the surface area of the sphere is proportional to the square of the radius, then the radius had gone up by a factor of $\sqrt{2}$. As volume is proportional to the cube of the radius, the volume has gone up by a factor of $(\sqrt{2})^3 = 2\sqrt{2}$.

8. Find the distance between the point with coordinates $(14, -2)$ and the line with equation $3x - 4y = 0$.

   (A) 4 **(B)** $4\sqrt{2}$ **(C)** $5\sqrt{2}$ **(D)** 8 **(E)** 10

   Solution: **(E)** The point of the line closest to $(14, -2)$ is the foot of the perpendicular from $(14, -2)$ dropped onto the line. This perpendicular has slope $-4/3$. Performing a run of -3 and a rise of 4 twice from the starting point of $(14, -2)$ gives the point $(8, 6)$. This is indeed on the line $3x - 4y = 0$ so the point $(8, 6)$ is this closest point. The distance between $(8, 6)$ and $(14, -2)$ is equal to $\sqrt{6^2 + 8^2} = 10$.

9. Zeke cycles steadily for 36 miles. If he had managed to go 3 mph faster, he would have taken one hour less for the trip. What was Zeke's actual speed in mph during this trip?

   (A) 9 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 21

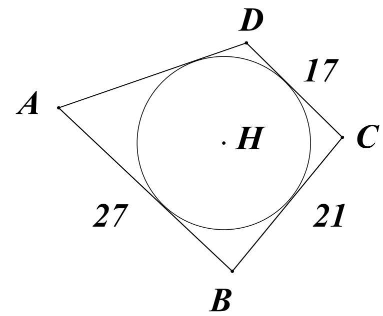
   Solution: **(A)** Let x be Zeke's actual speed and $x + 3$ the supposed speed. Then since time equals distance divided by speed we have

   $$\frac{36}{x} - \frac{36}{x+3} = 1$$

   Solving this equation yields $x = 9$.

10. A circle is inscribed in quadrilateral $ABCD$ as marked. Find the length of side $\overline{DA}$.

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27



Solution: **(A)** From the equality of tangents to a circle one can show that $AB + CD = BC + AD$ or $27 + 17 = 21 + AD$. Thus $AD = 44 - 21 = 23$.

11. How many lines of symmetry does a cube have?

(A) 4 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

Solution: **(E)** There are three such lines joining the centers of opposite faces, four joining opposite vertices, and six joining the midpoints of opposite edges.

12. In $\triangle ABC$, let D be the intersection point of the bisector of $\angle ABC$ and the bisector of $\angle BCA$. If $\angle CAB$ is 70 degrees, what is the measure of $\angle CDB$ in degrees?

(A) 35 **(B)** 55 **(C)** 105 **(D)** 125 **(E)** 140

Solution: **(D)** The angle measures of $\angle A$ and $\angle B$ must sum of $180 - 70$ or 110. So in $\triangle BCD$ the two angles at B and C must add to 55. Thus $\angle CDB = 180 - 55 = 125$.

13. A set of seven distinct positive integers has a mean of 13. Find the difference between the greatest possible median of these integers and the least possible median of these integers.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: **(D)** The sum of all seven integers is 91. The least possible median of 4 comes about from the set $\{1, 2, 3, 4, 5, 6, 70\}$. The greatest possible median of 19 occurs in the set $\{1, 2, 3, 19, 20, 21, 25\}$, among others. The number 20 cannot be the median as then the smallest possible sum would be $1+2+3+20+21+22+23 = 92$. The answer is $19-4 = 15$.

14. A line L in the coordinate plane has slope -2. Suppose the triangle with vertices given by the origin, the x-intercept of L, and the y-intercept of L has area 9. Then an equation for L could be

(A) $2x + y = 0$ **(B)** $2x + y = 4$ **(C)** $-2x + y = 6$
(D) $2x + y = 3$ **(E)** $2x + y = -6$

Solution: **(E)** Because the slope of the line has absolute value 2, then this right triangle must have legs of length a and $2a$. Thus $(1/2)a(2a) = a^2 = 9$ so that $a = 3$ or $a = -3$. If $a = 3$, then the line does not match any of the equations given. So $a = -3$ does and leads to $2x + y = -6$.

15. Each vertex of a cube is randomly colored red or blue with each color being equally likely. What is the probability that every pair of adjacent vertices have different colors?

(A) 0 **(B)** 1/128 **(C)** 1/64 **(D)** 1/32 **(E)** 1/2

Solution: **(B)** Consider a particular vertex of the cube. It does not matter which one or which color it is. Then the three vertices that are connected by an edge with this chose vertex must have opposite color to it. That happens with probabiltiy $1/2^3$ or $1/8$. These three newly colored vertices force 3 more vertices to be colored the same as the original vertex. This happens with probability $1/8$. Finally the last vertex is colored opposite those just determined, with probability $1/2$. So the final probability of all this happening is $(1/8)(1/8)(1/2) = 1/128$.

16. A semicircle lies in $\triangle EFG$ with diameter contained in $\overline{EG}$, and with $\overline{EF}$ and $\overline{GF}$ both tangent to it. If $EF = 12$, $FG = 15$, and $EG = 18$, what is the value of EC where C is the center of the semicircle?

(A) 6 **(B)** 6.5 **(C)** 7 **(D)** 7.5 **(E)** 8

Solution: **(E)** Note that $\overline{FC}$ is an angle bisector of $\angle EFG$. Let $x = EC$ with then $GC = 18 - x$. By the Angle Bisector Theorem, $x/12 = (18 - x)/15$, showing that $x = 8$ is correct.

17. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1 inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solution: **(A)** Let (i, j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2, 1)$. Withoug loss of generality we can assume that the color that went in $(2, 1)$ is the same as the color that went in $(1, 4)$. There arise two cases depending on the color for $(2, 4)$.

Case I. Color $(2, 4)$ the same as $(1, 1)$. Then the colors for the remaining squares of row 2 are determined. For coloring $(3, 1)$ and $(3, 2)$ there are 2 ways. This choice forces the color choices in the remaining squares of column 4. Likewise there are two choices for the last 2 squares of column 2, which force the choices for the last two squares of column three. Counting the previous choices Case I gives $4! \cdot 3 \cdot 2 \cdot 2$ or 288 choices.

Case II. Color $(2, 4)$ the same as either $(1, 2)$ or $(2, 2)$. This gives 2 choices and forces the colors for the remaining squares of row 2. For the last two squares of the first column there are 2 choices. One can check that this last choice forces the remaining 6 squares. So Case II gives altogether 288 choices as well, though it does not seem obvious why this should be so. I would be interested in an argument of why it must be so if it must be so. (What would happen if it where a 5 by 5 square with 5 colors? Is it also the square of 5!??)

Adding our two disjoint cases together we obtain $288 + 288$ or $576 = 24^2$.

18. In acute $\triangle ABC$, the altitude from A meets side $\overline{BC}$ at point D, the altitude from B meets side $\overline{AC}$ at point E, and the altitude from C meets side $\overline{AB}$ at point F. All three altitudes are concurrent at point H lying inside $\triangle ABC$. If $\angle BAC$ measures 58 degrees, then find the measure of $\angle BHC$ in degrees.

(A) 90 **(B)** 98 **(C)** 104 **(D)** 116 **(E)** 122

Solution: **(E)** By the Exterior Angle Theorem used twice,

$$\begin{aligned}\angle BHC &= \angle BHD + \angle CHD \\ &= (\angle HAB + \angle ABH) + (\angle HAC + \angle CAH) \\ &= (\angle HAB + \angle HAC) + (\angle ABE + \angle CAF) \\ &= \angle BAC + (90 - \angle BAC) + (90 - \angle BAC) \\ &= 180 - \angle BAC = 180 - 58 = 122\end{aligned}$$

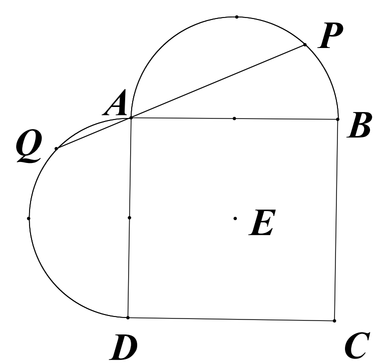
19. In how many ways can five distinct books be arranged in a bookcase with 3 shelves, each shelf capable of holding all five books?

(A) 19 **(B)** 120 **(C)** 360 **(D)** 840 **(E)** 2520

Solution: **(E)** First we count the number of ways to assign how many books each shelf gets. To the five books, add two dividers to distinguish the shelves, giving, we imagine, 7 blanks in a row. By choosing two of the blanks to be dividers (so the remaining blanks become books) we determine how many books go on each shelf. This can be done in $\binom{7}{2}$ or 21 ways. Next we can assign particular books to each of the five spots just determined in 5! or 120 ways. The total count of ways to arrange the books in the bookcase becomes $21 \cdot 120$ or 2520.

20. Semicircles are drawn on two sides of square $ABCD$ as shown. Point E is the center of the square, and points Q, A, and P are collinear with $QA = 4$ and $AP = 16$. Find QE.

(A) 12 **(B)** $10\sqrt{2}$ **(C)** $10\sqrt{3}$
(D) 15 **(E)** 20



Solution: **(B)** Complete semicircle APB into circle $APBE$. Then $\angle ABE = 45°$ because E is the center of square $ABCD$. But inscribed $\angle QPE$ intercepts the same arc as does inscribed $\angle ABE$ in this circle. So the two inscribed angles are congruent, giving $\angle QPE = 45°$. Likewise, $\angle PQE = 45°$. As $\triangle QPE$ is now seen to be an isosceles right triangle it follows quickly that a leg QE must have length $20/\sqrt{2}$ or $10\sqrt{2}$.

Spring 2010 McNabb GDCTM Contest
Level II Solutions

1. What is the area of the quadrilateral in the coordinate plane with vertices whose coordinates are (in order): $(0,0)$, $(7,1)$, $(4,4)$, and $(2,11)$?

   (A) 30 **(B)** 31 **(C)** 31.5 **(D)** 32 **(E)** 33

   Solution: **(A)** This can be done with the shoestring method. Setting up the determinants by going counterclockwise around the quadrilateral gives for the area

$$\frac{28+44-4-8}{2} = 60/2 = 30$$

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   Solution: **(C)** Notice that 24, 25, 26, and 27 cent postage can all be made by Ronald. Once 4 in a row are established like this, all higher amounts of postage is possible, simply by adding the right number of 4 cent stamps. Since 23 cents of postage cannot be made, 23 is then the largest such amount.

3. Find the distance between the point with coordinates $(14,-2)$ and the line with equation $3x-4y=0$.

   (A) 4 **(B)** $4\sqrt{2}$ **(C)** $5\sqrt{2}$ **(D)** 8 **(E)** 10

   Solution: **(E)** The point of the line closest to $(14,-2)$ is the foot of the perpendicular from $(14,-2)$ dropped onto the line. This perpendicular has slope $-4/3$. Performing a run of -3 and a rise of 4 twice from the starting point of $(14,-2)$ gives the point $(8,6)$. This is indeed on the line $3x-4y=0$ so the point $(8,6)$ is this closest point. The distance between $(8,6)$ and $(14,-2)$ is equal to $\sqrt{6^2+8^2}=10$.

4. Zeke cycles steadily for 36 miles. If he had managed to go 3 mph faster, he would have taken one hour less for the trip. What was Zeke's actual speed in mph during this trip?

   (A) 9 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 21

   Solution: **(A)** Let x be Zeke's actual speed and $x+3$ the supposed speed. Then since time equals distance divided by speed we have

$$\frac{36}{x} - \frac{36}{x+3} = 1$$

   Solving this equation yields $x=9$.

5. How many lines of symmetry does a cube have?

   (A) 4 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

   Solution: **(E)** There are three such lines joining the centers of opposite faces, four joining opposite vertices, and six joining the midpoints of opposite edges.

6. In $\triangle ABC$, let D be the intersection point of the bisector of $\angle ABC$ and the bisector of $\angle BCA$. If $\angle CAB$ is 70 degrees, what is the measure of $\angle CDB$ in degrees?

(A) 35 **(B)** 55 **(C)** 105 **(D)** 125 **(E)** 140

Solution: **(D)** The angle measures of $\angle A$ and $\angle B$ must sum of $180 - 70$ or 110. So in $\triangle BCD$ the two angles at B and C must add to 55. Thus $\angle CDB = 180 - 55 = 125$.

7. Hezy eats y yogurts every d days. How many yogurts does he eat in w weeks?

(A) $\frac{7yw}{d}$ **(B)** $\frac{7w}{yd}$ **(C)** $\frac{yd}{7w}$ **(D)** $7dwy$ **(E)** $\frac{7yd}{w}$

Solution: **(A)** In w weeks there are $7w/d$ blocks of d days, each block contributing y yogurts.

8. In throwing four fair cubical dice, what is the probability of obtaining two distinct doubles?

(A) $\frac{5}{72}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{3}{8}$

Solution: **(A)** There are $\binom{6}{2}$ ways of choosing which 2 numbers will appear as doubles, then $\binom{4}{2}$ ways of choosing which dice get which of those numbers. Multiply theses quantities, then divide by 6^4 to find the probability.

9. A set of seven distinct positive integers has a mean of 13. Find the difference between the greatest possible median of these integers and the least possible median of these integers.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: **(D)** The sum of all seven integers is 91. The least possible median of 4 comes about from the set $\{1, 2, 3, 4, 5, 6, 70\}$. The greatest possible median of 19 occurs in the set $\{1, 2, 3, 19, 20, 21, 25\}$, among others. The number 20 cannot be the median as then the smallest possible sum would be $1+2+3+20+21+22+23 = 92$. The answer is $19 - 4 = 15$.

10. The parabola $y = ax^2 + bx + c$ passes through the points $(-2, 3)$, $(2, -1)$, and $(6, 12)$. The value of the coefficient a equals

(A) 1/4 **(B)** 3/16 **(C)** 5/16 **(D)** 17/32 **(E)** 1/2

Solution: **(D)** Note that we can write $y = k_1(x+2)(x-2)+k_2(x-2)(x-6)+k_3(x+2)(x-6)$ and solve for the values of k_i. We obtain $k_1 = 3/8$, $k_2 = 3/32$ and $k_3 = 1/16$. Thus $a = k_1 + k_2 + k_3 = 17/32$.

11. The *centroid* of a triangle is the point of concurrence of its medians. In the $x - y$ plane point A has coordinates $(0, 0)$, point B has coordinates $(5, 15)$, and point C has coordinates $(13, 9)$. The line p passes through the point B and the centroid of $\triangle ABC$. Another point on line p is

(A) $(6, 9)$ **(B)** $(12, -2)$ **(C)** $(7, 1)$ **(D)** $(0, 43)$ **(E)** $(-4, -4)$

Solution: **(C)** The coordinates of the centroid are the arithmetical average (mean) of the coordinates of the three vertices. So $G = ((0 + 5 + 13)/3, (0 + 15 + 9)) = (6, 8)$. So the line p has slope -7 and equation $7x + y = 50$. Then $(7, 1)$ lies on line p while the others do not.

12. If $\sin x + \cos x = a$, then $\sin 2x$ equals

(A) $2a$ **(B)** $a^2 - 1$ **(C)** $1 - a^2$ **(D)** $a^2 + 1$ **(E)** $(a-1)^2$

Solution: **(B)** Square both sides and use the Pythagorean identity and the double angle identity for sine.

13. A line L in the coordinate plane has slope -2. Suppose the triangle with vertices given by the origin, the x-intercept of L, and the y-intercept of L has area 9. Then an equation for L could be

(A) $2x + y = 0$ **(B)** $2x + y = 4$ **(C)** $-2x + y = 6$
(D) $2x + y = 3$ **(E)** $2x + y = -6$

Solution: **(E)** Because the slope of the line has absolute value 2, then this right triangle must have legs of length a and $2a$. Thus $(1/2)a(2a) = a^2 = 9$ so that $a = 3$ or $a = -3$. If $a = 3$, then the line does not match any of the equations given. So $a = -3$ does and leads to $2x + y = -6$.

14. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1 inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solution: **(A)** Let (i, j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2, 1)$. Withoug loss of generality we can assume that the color that went in $(2, 1)$ is the same as the color that went in $(1, 4)$. There arise two cases depending on the color for $(2, 4)$.

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(A) 90 **(B)** 98 **(C)** 104 **(D)** 116 **(E)** 122

Solution: **(E)** By the Exterior Angle Theorem used twice,

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16. In how many ways can five distinct books be arranged in a bookcase with 3 shelves, each shelf capable of holding all five books?

(A) 19 **(B)** 120 **(C)** 360 **(D)** 840 **(E)** 2520

Solution: **(E)** First we count the number of ways to assign how many books each shelf gets. To the five books, add two dividers to distinguish the shelves, giving, we imagine, 7 blanks in a row. By choosing two of the blanks to be dividers (so the remaining blanks become books) we determine how many books go on each shelf. This can be done in $\binom{7}{2}$ or 21 ways. Next we can assign particular books to each of the five spots just determined in 5! or 120 ways. The total count of ways to arrange the books in the bookcase becomes $21 \cdot 120$ or 2520.

17. Two ferries start at the same instant from opposite banks of a river. They travel directly across the river. Each boat keeps its own constant speed, though one boat is faster than the other. In this first trip across they pass at a point 720 yards from the nearer bank. When reaching the opposite shore each boat remains exactly 10 minutes in its dock before heading back the other way. On this trip back the boats meet 400 yards from the other shore. How wide is the river (in yards)?

(A) 1040 **(B)** 1120 **(C)** 1520 **(D)** 1600 **(E)** 1760

Solution: **(E)** Let x be the distance the faster ferrie travels from its first departure until it meets the slower ferry coming the other way. At this point both ferries are 720 yards from the nearer shore, which is where the slower ferry must have departed. As the speeds of the ferries remain constant the ratio of the distances they travel in an equal time must remain the same. By drawing a careful diagram you can determine that from their first meeting until their second the faster ferry has travelled at total of $x + 320 + 720$ while the slower has travelled at total of $x + 400$. Thus

$$\frac{x}{720} = \frac{x + 320 + 720}{x + 400}$$

Thus $x^2 - 320x - 720(720 + 320) = (x - (720 + 320))(x + 720) = 0$. Since $x > 0$ we must have $x = 720 + 320 = 1040$. The width of the river is $x + 720 = 1040 + 720 = 1760$. There is a simpler way to do this problem. Try to find it! This problem was created in the nineteenth century by Sam Lloyd, the great American problem poser.

18. If p and q are integers and

$$p \log_{200} 5 + q \log_{200} 2 = 3$$

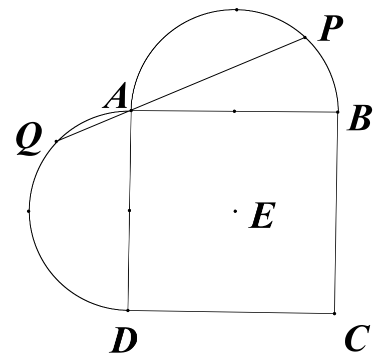
then determine the value of $p + q$.

(A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 20

Solution: **(C)** The left hand side of the given equation can be written as $\log_{200}(5^p \cdot 2^q)$ while the right hand side can be written $\log_{200} 200^3$. It follows from this that $5^p \cdot 2^q = 200^3 = 5^6 \cdot 2^9$ so $p = 6$ and $q = 9$. And $p + q = 6 + 9 = 15$.

19. Semicircles are drawn on two sides of square $ABCD$ as shown. Point E is the center of the square, and points Q, A, and P are collinear with $QA = 4$ and $AP = 16$. Find QE.

(A) 12 **(B)** $10\sqrt{2}$ **(C)** $10\sqrt{3}$
(D) 15 **(E)** 20



Solution: **(B)** Complete semicircle APB into circle $APBE$. Then $\angle ABE = 45°$ because E is the center of square $ABCD$. But inscribed $\angle QPE$ intercepts the same arc as does inscribed $\angle ABE$ in this circle. So the two inscribed angles are congruent, giving $\angle QPE = 45°$. Likewise, $\angle PQE = 45°$. As $\triangle QPE$ is now seen to be an isosceles right triangle it follows quickly that a leg QE must have length $20/\sqrt{2}$ or $10\sqrt{2}$.

20. In $\triangle ABC$, points D, E, and F are located on $\overline{BC}$, $\overline{AC}$, and $\overline{BA}$ respectively, so that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point P, the area of $\triangle BPD$ is 190, the area of $\triangle DPC$ is 380, and the area of $\triangle CPE$ is 418. Then the area of $\triangle APE$ is

(A) 121 **(B)** 143 **(C)** 242 **(D)** 319 **(E)** 330

Solution: **(C)** Use the method of points masses (barycentric coordinates). Let x be the area of $\triangle APE$. Place a mass of $418/x$ at A, a mass of 2 at B and a mass of 1 at C. So masses B and C balance at D, masses A and C at E and masses A and B at F. This put masses of 3 at D, $1 + 418/x$ at E, and $2 + 418/x$ at F. Let y be the area of $\triangle APF$ and z the area of $\triangle BPF$. Since point masses C and F balance at P we have $z/570 = 1/(2 + 418/x)$ so $z = 570/(2 + 418/x)$. Similarly since A and B balance at F, $y/z = 2/(418/x) = x/209$. By substitution $y = 570x/(209(2 + 418/x))$. Since masses at B and C balance at D we obtain

$$2\left(190 + \frac{570x}{209(2 + 418/x)} + \frac{570}{2 + 418/x}\right) = x + 380 + 418$$

which simplifies to the quadratic equation $38x^2 - 1254x - 1921964 = 0$. The quadratic factors as $(x - 242)(38x + 7942) = 0$. Thus $x = 242$ is the required area.

Spring 2010 McNabb GDCTM Contest
Level III Solutions

1. Amy is in a line of 40 people at the movies. Ahead of her stand 12 people. How many stand behind her?

   (A) 13 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 39

   Solution: **(B)**. Don't forget to count Amy herself. So $12 + 1 + 27 = 40$.

2. Abigail, Brice, and Carl all start out with some one-dollar bills in their pockets. In particular, Carl starts out with 4 one-dollar bills. Brice then gives half of his dollar bills to Abigail and the other half to Carl. Then Abigail gives half of her bills to Brice and the other half to Carl. Finally, Carl gives half of his bills to Abigail and keeps the rest. If now, at the end of these exchanges, Abigail, Brice, and Carl all have 8 one-dollar bills, how many such bills did Abigail have to begin with?

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

   Solution: **(E)** Work backwards. Just before the last exchange Abigail has zero dollar bills (because she has just given away all of her bills), Brice eight, and Carl sixteen. So working back one more exchange, Abigail would have 16 bills, Brice zero, and Carl eight. Since Carl started with four he must gain four on the first exchange, meaning that Abigail also gained four bills on the first exchange, implying she started with 12 bills.

3. Let a, b, c, and d be positive real numbers such that $abc = 12$, $bcd = 6$, $acd = 125$, and $abd = 3$. Then $abcd$ equals

   (A) 12 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

   Solution: **(C)** Multiply all the equations together and take the cube root of both sides.

4. Let $f(x)$ be a linear function for which $f(8) - f(1) = 11$. Then $f(41) - f(6)$ equals

   (A) 37 **(B)** 43 **(C)** 49 **(D)** 55 **(E)** 61

   Solution: **(D)** From the first equation the slope of the line must be $11/7$. Therefore $f(41) - f(6) = (11/7)(41 - 6) = (11/7)35 = 11 \cdot 5 = 55$.

5. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

   Solution: **(C)** Let the plums be numeraire, that is, assign plum equal to one. Let a pomegranate weigh x and a pineapple weigh y. Solve the system $3x + y = 16$, $4 + x = y$, for example, by substitution. We get $3x + 4 + x = 16$ or $x = 3$. Then $y = 7$. So three pineapples weigh 21 and it takes 7 pomegranates to weigh as much.

6. If p and q are integers and

$$p \log_{200} 5 + q \log_{200} 2 = 3$$

then determine the value of $p + q$.

(A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 20

Solution: **(C)** The left hand side of the given equation can be written as $\log_{200}(5^p \cdot 2^q)$ while the right hand side can be written $\log_{200} 200^3$. It follows from this that $5^p \cdot 2^q = 200^3 = 5^6 \cdot 2^9$ so $p = 6$ and $q = 9$. And $p + q = 6 + 9 = 15$.

7. For $z = a + bi$ a complex number, it's conjugate is $\overline{z} = a - bi$. Let S denote the set of all complex numbers z so that the real part of $1/\overline{z}$ equals one. Then set S is

(A) a line **(B)** a circle **(C)** a parabola
(D) the empty set **(E)** an hyperbola

Solution: **(B)** With notation as in the problem, $1/\overline{z} = z/(z\overline{z}) = (a+bi)/(a^2+b^2)$. Setting the real part of this equal to 1 we obtain $a/(a^2+b^2) = 1$ or $a^2-a+b^2 = 0$ or $(a-1/2)^2+b^2 = 1/4$, a circle.

8. If a is a multiple of 14 and b is a multiple of 21, then what is the largest integer that must be a factor of any integer of the form $9a + 8b$?

(A) 84 **(B)** 42 **(C)** 21 **(D)** 14 **(E)** 8

Solution: **(B)** The greatest common factor of 14 and 21 is 7. So $9a + 8b$ is certainly divisible by 7. With 9 being divisible by 3 and 21 being divisible by 3 then $9a + 8b$ is also divisible by 3. And since a is divisible by 2 and 8 is divisible by 2, then $9a + 8b$ is divisible by 2. We can be certain of no further divisibilities, the answer is $7 \cdot 3 \cdot 2$ or 42.

9. How many lines of symmetry does a cube have?

(A) 4 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

Solution: **(E)** There are three such lines joining the centers of opposite faces, four joining opposite vertices, and six joining the midpoints of opposite edges.

10. The parabola $y = ax^2 + bx + c$ passes through the points $(-2, 3)$, $(2, -1)$, and $(6, 12)$. The value of the coefficient a equals

(A) 1/4 **(B)** 3/16 **(C)** 5/16 **(D)** 17/32 **(E)** 1/2

Solution: **(D)** Note that we can write $y = k_1(x+2)(x-2)+k_2(x-2)(x-6)+k_3(x+2)(x-6)$ and solve for the values of k_i. We obtain $k_1 = 3/8$, $k_2 = 3/32$ and $k_3 = 1/16$. Thus $a = k_1 + k_2 + k_3 = 17/32$.

11. The *centroid* of a triangle is the point of concurrence of its medians. In the $x - y$ plane point A has coordinates $(0, 0)$, point B has coordinates $(5, 15)$, and point C has coordinates $(13, 9)$. The line p passes through the point B and the centroid of $\triangle ABC$. Another point on line p is

(A) $(6,9)$ **(B)** $(12,-2)$ **(C)** $(7,1)$ **(D)** $(0,43)$ **(E)** $(-4,-4)$

Solution: **(C)** The coordinates of the centroid are the arithmetical average (mean) of the coordinates of the three vertices. So $G = ((0+5+13)/3, (0+15+9)) = (6,8)$. So the line p has slope -7 and equation $7x+y=50$. Then $(7,1)$ lies on line p while the others do not.

12. If $\sin x + \cos x = a$, then $\sin 2x$ equals

(A) $2a$ **(B)** a^2-1 **(C)** $1-a^2$ **(D)** a^2+1 **(E)** $(a-1)^2$

Solution: **(B)** Square both sides and use the Pythagorean identity and the double angle identity for sine.

13. In throwing four fair cubical dice, what is the probability of obtaining two distinct doubles?

(A) $\dfrac{5}{72}$ **(B)** $\dfrac{7}{36}$ **(C)** $\dfrac{1}{5}$ **(D)** $\dfrac{5}{16}$ **(E)** $\dfrac{3}{8}$

Solution: **(A)** There are $\binom{6}{2}$ ways of choosing which 2 numbers will appear as doubles, then $\binom{4}{2}$ ways of choosing which dice get which of those numbers. Multiply theses quantities, then divide by 6^4 to find the probability.

14. The function $f(x) = \dfrac{x+\sqrt{x^2+8}}{2}$ has domain $(-\infty,\infty)$ and on this domain is invertible. It's inverse function has the form $f^{-1}(x) = ax + \dfrac{b}{x}$ for some constants a and b. What is the value of $a+b$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

Solution: **(A)** Note that the value of $f^{-1}(1)$ is the same as $a+b$. Since $f(-1)=1$, then $f^{-1}(1) = -1 = a+b$.

15. If $x + \dfrac{1}{x} = 3$ then what is the value of $\dfrac{x^4+1}{x^2}$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution: **(D)** Square both sides of the first equation yielding $x^2+2+1/x^2 = 9$ or $x^2 + 1/x^2 = 9-2 = 7 = \dfrac{x^4+1}{x^2}$

16. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solution: **(A)** Let (i,j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2,1)$. Withoug loss of generality we can assume that the color that went in $(2,1)$ is the same as the color that went in $(1,4)$. There arise two cases depending on the color for $(2,4)$.

Case I. Color $(2,4)$ the same as $(1,1)$. Then the colors for the remaining squares of row 2 are determined. For coloring $(3,1)$ and $(3,2)$ there are 2 ways. This choice forces the color choices in the remaining squares of column 4. Likewise there are two choices for the last 2 squares of column 2, which force the choices for the last two squares of column three. Counting the previous choices Case I gives $4! \cdot 3 \cdot 2 \cdot 2$ or 288 choices.

Case II. Color $(2,4)$ the same as either $(1,2)$ or $(2,2)$. This gives 2 choices and forces the colors for the remaining squares of row 2. For the last two squares of the first column there are 2 choices. One can check that this last choice forces the remaining 6 squares. So Case II gives altogether 288 choices as well, though it does not seem obvious why this should be so. I would be interested in an argument of why it must be so if it must be so. (What would happen if it where a 5 by 5 square with 5 colors? Is it also the square of 5!??)

Adding our two disjoint cases together we obtain $288 + 288$ or $576 = 24^2$.

17. A semicircle lies in $\triangle EFG$ with diameter contained in $\overline{EG}$, and with $\overline{EF}$ and $\overline{GF}$ both tangent to it. If $EF = 12$, $FG = 15$, and $EG = 18$, what is the value of EC where C is the center of the semicircle?

    (A) 6 **(B)** 6.5 **(C)** 7 **(D)** 7.5 **(E)** 8

    Solution: **(E)** Note that $\overline{FC}$ is an angle bisector of $\angle EFG$. Let $x = EC$ with then $GC = 18 - x$. By the Angle Bisector Theorem, $x/12 = (18 - x)/15$, showing that $x = 8$ is correct.

18. A cubic polynomial $P(x)$ satisfies $P(1) = 1$, $P(2) = 3$, $P(3) = 5$, and $P(4) = 6$. Then the value $P(7)$ must equal

    (A) 10 **(B)** 7 **(C)** 0 **(D)** -3 **(E)** -7

    Solution: **(E)** Note that the first three values of P listed in the problem are in arithemetic progression. This means they follow a linear function, in fact $2x - 1$. So let's take away this to arrive at a nicer polynomial $Q(x)$ by letting $Q(x) = P(x) - (2x - 1)$. Thus $Q(1) = Q(2) = Q(3) = 0$ and $Q(4) = -1$. The cubic polynomial $Q(x)$ has roots of $1, 2$ and 3 so by the Factor Theorem must be of the form $Q(x) = a(x-1)(x-2)(x-3)$ for some constant a. From $Q(4) = -1$ follows $a = -1/6$. Now we have $P(x) = (-1/6)(x-1)(x-2)(x-3) + 2x - 1$ and $P(7) = (-1/6)(6)(5)(4) + 14 - 1 = -7$.

19. Hezy tosses a fair cubical die until each number 1 through 6 has come up at least once. On average, Hezy requires X tosses to do this. The real number X is closest to

    (A) 12.8 **(B)** 14.7 **(C)** 16.3 **(D)** 17.2 **(E)** 19.5

    Solution: **(B)** Let the die be thrown n times. By the Principle of Inclusion/Exclusion there are

$$6^n - 6 \cdot 5^n + 15 \cdot 4^n - 20 \cdot 3^n + 15 \cdot 2^n - 6 \cdot 1^n$$

ways in which none of the six values will be missing. Thus the probability f_n of failing to reach the goal after n tosses is

$$f_n = 1 - \frac{(6^n - 6 \cdot 5^n + 15 \cdot 4^n - 20 \cdot 3^n + 15 \cdot 2^n - 6 \cdot 1^n)}{6^n}$$
$$= 6\left(\frac{5}{6}\right)^n - 15\left(\frac{4}{6}\right)^n + 20\left(\frac{3}{6}\right)^n - 15\left(\frac{2}{6}\right)^n + 6\left(\frac{1}{6}\right)^n$$

The expected number of tosses is $1 + f_1 + f_2 + f_3 + \cdots + f_n + \cdots$ Summing these five infinite geometric series and adding one does give the expected number of tosses as 14.7.

20. In $\triangle ABC$, points D, E, and F are located on $\overline{BC}$, $\overline{AC}$, and $\overline{BA}$ respectively, so that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point P, the area of $\triangle BPD$ is 190, the area of $\triangle DPC$ is 380, and the area of $\triangle CPE$ is 418. Then the area of $\triangle APE$ is

(A) 121 **(B)** 143 **(C)** 242 **(D)** 319 **(E)** 330

Solution: **(C)** Use the method of points masses (barycentric coordinates). Let x be the area of $\triangle APE$. Place a mass of $418/x$ at A, a mass of 2 at B and a mass of 1 at C. So masses B and C balance at D, masses A and C at E and masses A and B at F. This put masses of 3 at D, $1+418/x$ at E, and $2+418/x$ at F. Let y be the area of $\triangle APF$ and z the area of $\triangle BPF$. Since point masses C and F balance at P we have $z/570 = 1/(2+418/x)$ so $z = 570/(2+418/x)$. Similarly since A and B balance at F, $y/z = 2/(418/x) = x/209$. By substitution $y = 570x/(209(2+418/x))$. Since masses at B and C balance at D we obtain

$$2\left(190 + \frac{570x}{209(2+418/x)} + \frac{570}{2+418/x}\right) = x + 380 + 418$$

which simplifies to the quadratic equation $38x^2 - 1254x - 1921964 = 0$. The quadratic factors as $(x-242)(38x+7942) = 0$. Thus $x = 242$ is the required area.

Spring 2010 McNabb GDCTM Contest
Level IV Solutions

1. Amy is in a line of 40 people at the movies. Ahead of her stand 12 people. How many stand behind her?

   (A) 13 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 39

2. Let a, b, c, and d be positive real numbers such that $abc = 12$, $bcd = 6$, $acd = 125$, and $abd = 3$. Then $abcd$ equals

   (A) 12 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

   Solutions: **(C)** Multiply all the equations together and take the cube root of both sides.

3. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

   Solutions: **(C)** Let the plums be numeraire, that is, assign plum equal to one. Let a pomegranate weigh x and a pineapple weigh y. Solve the system $3x + y = 16$, $4 + x = y$, for example, by substitution. We get $3x + 4 + x = 16$ or $x = 3$. Then $y = 7$. So three pineapples weigh 21 and it takes 7 pomegranates to weigh as much.

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   (A) a line **(B)** a circle **(C)** a parabola
   (D) the empty set **(E)** an hyperbola

   Solutions: **(B)** With notation as in the problem, $1/\overline{z} = z/(z\overline{z}) = (a + bi)/(a^2 + b^2)$. Setting the real part of this equal to 1 we obtain $a/(a^2 + b^2) = 1$ or $a^2 - a + b^2 = 0$ or $(a - 1/2)^2 + b^2 = 1/4$, a circle.

5. How many lines of symmetry does a cube have?

   (A) 4 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

   Solutions: **(E)** There are three such lines joining the centers of opposite faces, four joining opposite vertices, and six joining the midpoints of opposite edges.

6. If $\sin x + \cos x = a$, then $\sin 2x$ equals

   (A) $2a$ **(B)** $a^2 - 1$ **(C)** $1 - a^2$ **(D)** $a^2 + 1$ **(E)** $(a - 1)^2$

   Solutions: **(B)** Square both sides and use the Pythagorean identity and the double angle identity for sine.

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(A) $\frac{5}{72}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{3}{8}$

Solutions: **(A)** There are $\binom{6}{2}$ ways of choosing which 2 numbers will appear as doubles, then $\binom{4}{2}$ ways of choosing which dice get which of those numbers. Multiply theses quantities, then divide by 6^4 to find the probability.

8. The function $f(x) = \dfrac{x+\sqrt{x^2+8}}{2}$ has domain $(-\infty, \infty)$ and on this domain is invertible. It's inverse function has the form $f^{-1}(x) = ax + \dfrac{b}{x}$ for some constants a and b. What is the value of $a+b$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

Solutions: **(A)** Note that the value of $f^{-1}(1)$ is the same as $a+b$. Since $f(-1) = 1$, then $f^{-1}(1) = -1 = a+b$.

9. A regular pentagon has each edge of length 2. Its area is closest to

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solutions: **(D)** Divide the pentagon into five congruent isosceles triangles by connecting each vertex to the center. Each of these triangles has an area equal to $(1/2)(2)\cot 36° \approx 1.376$. The pentagon has area approximately $5(1.376)$ or closest to 7.

10. Let k be a positive constant and let f be a continuous function on the interval $[-k, k]$. If $\int_{-k}^{k} f(x)\,dx = a$ then $\int_{-1}^{1} f(kx)\,dx$ equals

(A) a **(B)** ak **(C)** 1 **(D)** $\frac{k}{a}$ **(E)** $\frac{a}{k}$

Solutions: **(E)** Make the change of variable $u = k/x$, $du = (1/k)dx$. Thus $a = k\int_{-1}^{1} f(ku)\,du$. Divide through by k and rename u as x.

11. Let f be continuously differentiable on the interval $[0, \pi]$. If $f(0) = 0$ and $f(\pi) = 0$, then

$$\int_0^\pi f(x)f'(x)\,dx$$

equals

(A) $-\pi$ **(B)** 0 **(C)** 1 **(D)** $\pi/2$ **(E)** cannot be uniquely determined

Solutions: **(B)** Make the change of variable $u = f(x)$, $du = f'(x)\,dx$. The original integral becomes $\int_0^0 u\,du = 0$.

12. Suppose for every positive x that

$$xe^x = e + \int_1^{x^3} f(t)\,dt$$

Find the value of $f(8)$.

(A) $e/4$ **(B)** e^2 **(C)** $e^2/4$ **(D)** $3e^2$ **(E)** 6

Solutions: **(C)** Use the fundamental theorem of calculus and the chain rule to differentiate both sides with respect to x. This gives $e^x + xe^x = 3x^2 f(x^3)$. Then put in $x = 2$. Note that the original equation does read true if $x = 1$, a necessary condition for there to exist a solution to this integral equation.

13. Find the area inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

(A) 5π **(B)** 6π **(C)** $25\pi/4$ **(D)** 7π **(E)** 8π

Solutions: **(B)** The area of an ellipse of semi-axes a and b is πab as can be found by integration. Here the area is $\pi \cdot 2 \cdot 3 = 6\pi$.

14. Let $\sum_{n=1}^{\infty} a_n$ be a positive term convergent series. Which of the following series must converge?

$$\text{I.} \quad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

$$\text{II.} \quad \sum_{n=1}^{\infty} \sqrt{a_n}$$

$$\text{III.} \quad \sum_{n=1}^{\infty} a_n^2$$

(A) I only **(B)** II only **(C)** III only **(D)** II and III only **(E)** I, II, and III

Solutions: **(C)** Since the original series converges, $\lim_{n\to\infty} a_n = 0$, so the terms in series I go to infinity and hence series I diverges. Series II does not always converges as consideration of the example $a_n = 1/n^2$ shows. Series III converges always since eventually $a_n^2 < a_n$ and the comparison test applies.

15. Let $I_n = \int_0^1 x^n e^{-x}\,dx$, where n is a positive integer. Which of the following is true?

$$\begin{array}{ll} \text{I.} & I_n = -e^{-1} + nI_{n-1} \quad \text{for } n \geq 2 \\ \text{II.} & I_n = n! - [en!]e^{-1} \quad \text{for } n \geq 2 \\ \text{III.} & \lim\limits_{n\to\infty} I_n = 0 \end{array}$$

where the notation $[x]$ stands for the greatest integer less than or equal to x.

(A) I only **(B)** II only **(C)** III only **(D)** I and III only **(E)** I, II, and III

Solutions: **(E)** Condition I holds by carrying out an integration by parts with $u = x^n$ and $dv = e^{-x}\,dx$. Condition II holds by an induction argument. Condition III holds because x_n tends to 0 as $x \to infty$ for all x in the interval $[0, 1)$ and the functions x_n are all bounded between 0 and 1 on the interval $[0, 1]$.

16. What is the coefficient of x^{10} in the expansion of

$$(1 + x)(1 + x^2)(1 + x^3)\cdots(1 + x^{10})$$

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 32

Solutions: **(B)** Using a single positive power of x (versus all 1's) there is one way. Using two positive distinct powers of x there are four ways; three positive distinct powers four ways, and four positive distinct powers one way. Thus $1 + 4 + 4 + 1 = 10$ ways altogether.

17. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solutions: **(A)** Let (i, j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2, 1)$. Withoug loss of generality we can assume that the color that went in $(2, 1)$ is the same as the color that went in $(1, 4)$. There arise two cases depending on the color for $(2, 4)$.

Case I. Color $(2, 4)$ the same as $(1, 1)$. Then the colors for the remaining squares of row 2 are determined. For coloring $(3, 1)$ and $(3, 2)$ there are 2 ways. This choice forces the color choices in the remaining squares of column 4. Likewise there are two choices for the last 2 squares of column 2, which force the choices for the last two squares of column three. Counting the previous choices Case I gives $4! \cdot 3 \cdot 2 \cdot 2$ or 288 choices.

Case II. Color $(2, 4)$ the same as either $(1, 2)$ or $(2, 2)$. This gives 2 choices and forces the colors for the remaining squares of row 2. For the last two squares of the first column there are 2 choices. One can check that this last choice forces the remaining 6 squares. So Case II gives altogether 288 choices as well, though it does not seem obvious why this should be so. I would be interested in an argument of why it must be so if it must be so. (What would happen if it where a 5 by 5 square with 5 colors? Is it also the square of 5!??)

Adding our two disjoint cases together we obtain $288 + 288$ or $576 = 24^2$.

18. A cubic polynomial $P(x)$ satisfies $P(1) = 1$, $P(2) = 3$, $P(3) = 5$, and $P(4) = 6$. Then the value $P(7)$ must equal

(A) 10 **(B)** 7 **(C)** 0 **(D)** -3 **(E)** -7

Solutions: **(E)** Note that the first three values of P listed in the problem are in arithemetic progression. This means they follow a linear function, in fact $2x-1$. So let's take away this to arrive at a nicer polynomial $Q(x)$ by letting $Q(x) = P(x) - (2x-1)$. Thus $Q(1) = Q(2) = Q(3) = 0$ and $Q(4) = -1$. The cubic polynomial $Q(x)$ has roots of $1, 2$ and 3 so by the Factor Theorem must be of the form $Q(x) = a(x-1)(x-2)(x-3)$ for some constant a. From $Q(4) = -1$ follows $a = -1/6$. Now we have $P(x) = (-1/6)(x-1)(x-2)(x-3) + 2x - 1$ and $P(7) = (-1/6)(6)(5)(4) + 14 - 1 = -7$.

19. Hezy tosses a fair cubical die until each number 1 through 6 has come up at least once. On average, Hezy requires X tosses to do this. The real number X is closest to

(A) 12.8 **(B)** 14.7 **(C)** 16.3 **(D)** 17.2 **(E)** 19.5

Solutions: **(B)** Let the die be thrown n times. By the Principle of Inclusion/Exclusion there are

$$6^n - 6\cdot 5^n + 15\cdot 4^n - 20\cdot 3^n + 15\cdot 2^n - 6\cdot 1^n$$

ways in which none of the six values will be missing. Thus the probability f_n of failing to reach the goal after n tosses is

$$\begin{aligned} f_n &= 1 - \frac{(6^n - 6\cdot 5^n + 15\cdot 4^n - 20\cdot 3^n + 15\cdot 2^n - 6\cdot 1^n)}{6^n} \\ &= 6\left(\frac{5}{6}\right)^n - 15\left(\frac{4}{6}\right)^n + 20\left(\frac{3}{6}\right)^n - 15\left(\frac{2}{6}\right)^n + 6\left(\frac{1}{6}\right)^n \end{aligned}$$

The expected number of tosses is $1 + f_1 + f_2 + f_3 + \cdots + f_n + \cdots$ Summing these five infinite geometric series and adding one does give the expected number of tosses as 14.7.

20. In $\triangle ABC$, points D, E, and F are located on $\overline{BC}$, $\overline{AC}$, and $\overline{BA}$ respectively, so that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point P, the area of $\triangle BPD$ is 190, the area of $\triangle DPC$ is 380, and the area of $\triangle CPE$ is 418. Then the area of $\triangle APE$ is

(A) 121 **(B)** 143 **(C)** 242 **(D)** 319 **(E)** 330

Solutions: **(C)** Use the method of points masses (barycentric coordinates). Let x be the area of $\triangle APE$. Place a mass of $418/x$ at A, a mass of 2 at B and a mass of 1 at C. So masses B and C balance at D, masses A and C at E and masses A and B at F. This put masses of 3 at D, $1 + 418/x$ at E, and $2 + 418/x$ at F. Let y be the area of $\triangle APF$ and z the area of $\triangle BPF$. Since point masses C and F balance at P we have $z/570 = 1/(2 + 418/x)$ so $z = 570/(2 + 418/x)$. Similarly since A and B balance at F, $y/z = 2/(418/x) = x/209$. By substitution $y = 570x/(209(2 + 418/x))$. Since masses at B and C balance at D we obtain

$$2\left(190 + \frac{570x}{209(2 + 418/x)} + \frac{570}{2 + 418/x}\right) = x + 380 + 418$$

which simplifies to the quadratic equation $38x^2 - 1254x - 1921964 = 0$. The quadratic factors as $(x - 242)(38x + 7942) = 0$. Thus $x = 242$ is the required area.

Spring 2010 McNabb GDCTM Contest
Level J1 Solutions

1. How many diagonals does a hexagon have?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

   Solution: **(E).** There are six ways to choose a first vertex and then three ways to choose the second one. So 18. But there is no such thing as a first and second endpoint of a diagonal so divide by 2! to get the answer of 9.

2. Amy is in a line of 40 people at the movies. Ahead of her stand 12 people. How many stand behind her?

   (A) 13 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 39

   Solution: **(B).** Don't forget to count Amy herself. So $12 + 1 + 27 = 40$.

3. What is the smallest positive integer which has the five smallest primes as factors?

   (A) 209 **(B)** 210 **(C)** 2310 **(D)** 15015 **(E)** 100000

   Solution: **(C).** Remember that the smallest prime is 2. So $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$.

4. At the post office Amy spent a total of \$ 5.00 to buy some 43 cent stamps and some 5 cent stamps. How many 5 cent stamps could she have bought?

   (A) 7 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 14

   Solution: **(E).** She must purchase the 43 cent stamps in multiples of 5. If she bought just 5, the number of 5 cent stamps is too great to appear in the list. Try 10 43 cent stamps. This leaves her 70 cents short, which take 14 five cent stamps.

5. The average of two positive numbers is equal to twice the smaller of the two numbers. How many times greater is the larger number than the smaller?

   (A) 1.5 times **(B)** 2 times **(C)** 2.5 times **(D)** 3 times **(E)** 4 times

   Solution: **(D).** Let the larger number be b and the smaller a. Then $(a + b)/2 = 2a$ or $a + b = 4a$ or $b = 3a$.

6. A kangaroo is 200 feet from a rabbit, when the kangaroo starts chasing the rabbit. Both immediately start hopping in the same direction. For each 13 foot leap of the kangaroo the rabbit takes two 4 foot leaps. From the time the chase began until the rabbit is caught, how many leaps did the rabbit take?

   (A) 20 **(B)** 40 **(C)** 80 **(D)** 100 **(E)** 200

   Solution: **(C).** The difference between 13 and twice 4 is 5. Five goes into 200 forty times, so the kangaroo must leap forty times and the rabbit twice that, or eighty times.

7. When three distinct numbers from the set $\{9, 8, -2, -4, -5\}$ are multiplied, the largest possible product is

   (A) 64 **(B)** 90 **(C)** 160 **(D)** 180 **(E)** 360

   Solution: **(D).** Since there are fewer than three positive numbers in the set, then two negative numbers must be used. So the greatest product is $9(-4)(-5)$ or 180.

8. The Elm school girls' basketball team has 11 girls on the team. The team will play 22 games this season, with each game lasting 32 minutes. The coach arranges for each girl to have the same total playing time by the end of the season. How many total minutes playing time would each girl end up with at the end of the season?

   (A) 64 **(B)** 128 **(C)** 200 **(D)** 320 **(E)** 440

   Solution: **(D)**. Since five players are on a basketball team during play, the total player minutes for the entire season amounts to $22 \cdot 32 \cdot 5$ minutes. Dividing this by 11 gives $2 \cdot 32 \cdot 5$ or 320 minutes.

9. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

   Solution: **(C)** Let the plums be numeraire, that is, assign plum equal to one. Let a pomegranate weigh x and a pineapple weigh y. Solve the system $3x + y = 16$, $4 + x = y$, for example, by substitution. We get $3x + 4 + x = 16$ or $x = 3$. Then $y = 7$. So three pineapples weigh 21 and it takes 7 pomegranates to weigh as much.

10. A box contains 10 red marbles, 11 blue marbles, and 12 green marbles. What is the fewest number of marbles you must pull out of the box to be sure of getting at least 5 of the same color?

    (A) 5 **(B)** 10 **(C)** 13 **(D)** 26 **(E)** 28

    Solution: **(C)**. If 4 of each color are taken this gives 12 total marbles, without 5 of any one color. Once the very next marble is chosen it must put one of the colors over to 5.

11. What is the area of the quadrilateral in the coordinate plane with vertices whose coordinates are (in order): $(0, 0)$, $(7, 1)$, $(4, 4)$, and $(2, 11)$?

    (A) 30 **(B)** 31 **(C)** 31.5 **(D)** 32 **(E)** 33

    Solution: **(A)** This can be done with the shoestring method. Setting up the determinants by going counterclockwise around the quadrilateral gives for the area

    $$\frac{28 + 44 - 4 - 8}{2} = 60/2 = 30$$

12. Ronald has an unlimited number of 5 cent and 7 cent stamps. What is the largest amount of postage (in cents) that he cannot make with these stamps?

    (A) 16 **(B)** 22 **(C)** 23 **(D)** 79 **(E)** 99

    Solution: **(C)** Notice that 24, 25, 26, and 27 cent postage can all be made by Ronald. Once 4 in a row are established like this, all higher amounts of postage is possible, simply by adding the right number of 4 cent stamps. Since 23 cents of postage cannot be made, 23 is then the largest such amount.

13. How many elements are in the set $\{7, 11, 15, 19, \cdots, 403\}$?

(A) 99 **(B)** 100 **(C)** 101 **(D)** 102 **(E)** 397

Solution: **(B)** Subtract three from each element of the set and then divide each of those numbers by 4 to obtain the set $\{1, 2, 3, \cdots, 100\}$, a set having the same number of elements as the original and also clearly having 100 elements.

14. Abigail, Brice, and Carl all start out with some one-dollar bills in their pockets. In particular, Carl starts with 4 one-dollar bills. Brice then gives half of his dollar bills to Abigail and the other half to Carl. Then Abigail gives half of her bills to Brice and the other half to Carl. Finally, Carl gives half of his bills to Abigail and keeps the rest. If now, at the end of these exchanges, Abigail, Brice, and Carl all have 8 one-dollar bills, how many such bills did Abigail have to begin with?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Solution: **(E)** Work backwards. Just before the last exchange Abigail has zero dollar bills (because she has just given away all of her bills), Brice eight, and Carl sixteen. So working back one more exchange, Abigail would have 16 bills, Brice zero, and Carl eight. Since Carl started with four he must gain four on the first exchange, meaning that Abigail also gained four bills on the first exchange, implying she started with 12 bills.

15. Farmer Ben sells thimbleberry jam in cylindrical jars. The company that supplies his jars is discontinuing the size jar he is currently using. Farmer Ben will have to order jars that have a diameter 10% less than his current jar. To maintain the volume of his current jar, the new jars he orders should have a height what percent greater than the current ones? Answer to the nearest percent.

(A) 11 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 124

Solution: **(B)** A cylinder with radius r and height h has volume $\pi r^2 h$. Decreasing the diameter by 10% also decreases the radius by 10%, so that r would become $.9r$. Since $.9^2 = .81$ it means that h must by multiplied by $1/.81 \approx 1.234567$, which is, to the nearest percent, a 23 percent increase in h.

16. Let a, b, c, and d be positive real numbers such that $abc = 12$, $bcd = 6$, $acd = 125$, and $abd = 3$. Then $abcd$ equals

(A) 12 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

Solution: **(C)** Multiply all the equations together and take the cube root of both sides.

17. In throwing four fair cubical dice, what is the probability of obtaining two distinct doubles?

(A) $\frac{5}{72}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{3}{8}$

Solution: **(A)** There are $\binom{6}{2}$ ways of choosing which 2 numbers will appear as doubles, then $\binom{4}{2}$ ways of choosing which dice get which of those numbers. Multiply theses quantities, then divide by 6^4 to find the probability.

18. If a is a multiple of 14 and b is a multiple of 21, then what is the largest integer that must be a factor of any integer of the form $9a + 8b$?

(A) 84 **(B)** 42 **(C)** 21 **(D)** 14 **(E)** 8

Solution: **(B)** The greatest common factor of 14 and 21 is 7. So $9a + 8b$ is certainly divisible by 7. With 9 being divisible by 3 and 21 being divisible by 3 then $9a + 8b$ is also divisible by 3. And since a is divisible by 2 and 8 is divisible by 2, then $9a + 8b$ is divisible by 2. We can be certain of no further divisibilities, the answer is $7 \cdot 3 \cdot 2$ or 42.

19. Each vertex of a cube is randomly colored red or blue with each color being equally likely. What is the probability that every pair of adjacent vertices have different colors?

(A) 0 **(B)** 1/128 **(C)** 1/64 **(D)** 1/32 **(E)** 1/2

Solution: **(B)** Consider a particular vertex of the cube. It does not matter which one or which color it is. Then the three vertices that are connected by an edge with this chose vertex must have opposite color to it. That happens with probabiltiy $1/2^3$ or $1/8$. These three newly colored vertices force 3 more vertices to be colored the same as the original vertex. This happens with probability $1/8$. Finally the last vertex is colored opposite those just determined, with probability $1/2$. So the final probability of all this happening is $(1/8)(1/8)(1/2) = 1/128$.

20. The value of the expression

$$(1 - (2 - (3 - (4 - (5 - (\cdots - (n)))) \cdots)$$

for n a positive even integer is equal to

(A) $-n$ **(B)** $-n/2$ **(C)** 0 **(D)** $n/2$ **(E)** $n - 3$

Solution: **(B)** Removing parentheses when n is even results in $1-2+3-4+5-6+\cdots+(n-1)-n$. Grouping in pairs as $(1-2)+(3-4)+\cdots(n-1-n) = -1+(-1)+(-1)+\cdots+(-1)$ shows that there are $n/2$ negative ones being added. Thus they add to $-n/2$.

21. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1 inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solution: **(A)** Let (i, j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2, 1)$. Withoug loss of generality we can assume that the color that went in $(2, 1)$ is the same as the color that went in $(1, 4)$. There arise two cases depending on the color for $(2, 4)$.

Case I. Color $(2, 4)$ the same as $(1, 1)$. Then the colors for the remaining squares of row 2 are determined. For coloring $(3, 1)$ and $(3, 2)$ there are 2 ways. This choice forces the color choices in the remaining squares of column 4. Likewise there are two choices for the last 2 squares of column 2, which force the choices for the last two squares of column three. Counting the previous choices Case I gives $4! \cdot 3 \cdot 2 \cdot 2$ or 288 choices.

Case II. Color $(2, 4)$ the same as either $(1, 2)$ or $(2, 2)$. This gives 2 choices and forces the colors for the remaining squares of row 2. For the last two squares of the first column there are 2 choices. One can check that this last choice forces the remaining 6 squares. So Case II gives altogether 288 choices as well, though it does not seem obvious why this should be so. I would be interested in an argument of why it must be so if it must be so. (What would happen if it where a 5 by 5 square with 5 colors? Is it also the square of 5!??)

Adding our two disjoint cases together we obtain $288 + 288$ or $576 = 24^2$.

22. The sequence $1, 3, 6, 10, 15, 21, \cdots$ is called the sequence of *triangular* numbers, because those numbers of dots can be arranged to form equilateral triangles. For instance, the triangular number 10 occurs in the set-up of bowling pins. What is the smallest triangular number greater than 1000?

    (A) 2^{10} **(B)** 1035 **(C)** 1021 **(D)** 1010 **(E)** 1006

    Solution: **(B)**. Let t_n denote the nth triangular number. Then $t_n = 1+2+3+\cdots+n = n(n+1)/2$. For $n(n+1)/2 \approx 1000$, $n^2 \approx 2000$. So we try values of n around $\sqrt{2000} \approx 44.7$. And $t_{44} = 44 \cdot 45/2 = 990$ and $t_{45} = 45 \cdot 46/2 = 1035$. So it is 1035.

23. In how many ways can five distinct books be arranged in a bookcase with 3 shelves, each shelf capable of holding all five books?

    (A) 19 **(B)** 120 **(C)** 360 **(D)** 840 **(E)** 2520

    Solution: **(E)** First we count the number of ways to assign how many books each shelf gets. To the five books, add two dividers to distinguish the shelves, giving, we imagine, 7 blanks in a row. By choosing two of the blanks to be dividers (so the remaining blanks become books) we determine how many books go on each shelf. This can be done in $\binom{7}{2}$ or 21 ways. Next we can assign particular books to each of the five spots just determined in 5! or 120 ways. The total count of ways to arrange the books in the bookcase becomes $21 \cdot 120$ or 2520.

24. The probability that Gerald wins any given game of HORSE is 3/5. Next Saturday, Gerald will play exactly five games of HORSE. What is the probability that he will win exactly three of them?

    (A) $\dfrac{108}{3125}$ **(B)** $\dfrac{3}{5}$ **(C)** $\dfrac{216}{625}$ **(D)** $\dfrac{9}{25}$ **(E)** 1

    Solution: **(C)** There are $\binom{5}{3}$ ways to choose which of the three games Gerald wins. For each of these choices, the probability it happens that way comes to $(3/5)^3(2/5)^2$. The probability Gerald does win exactly three games is $\binom{5}{3}(3/5)^3(2/5)^2 = 216/625$.

25. Seven stacks are made each consisting of seven half-dollar coins. One entire stack is made of counterfeit coins. All other stacks have true half-dollars. You know the weight of true half-dollars in grams. And each counterfeit half-dollar weighs exactly one gram more than the true coin. You can weigh the coins or any subset of them on a digital scale (similar to a regular bathroom scale) which outputs in grams. What is the minimum number of weighings needed to determine which stack is the counterfeit one?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 7

    Solution: **(A)** Surprisingly, this can be done in one weighing! Number the stacks from 1 to 7. Take one coin from stack #1 , two coins from stack #2, etc, all the way to all seven coins from stack # 7. Put all these coins on the scale. Their weight will be k grams over the weight of that many genuine coins. This can only occur if stack # k is the counterfeit one.

Spring 2010 McNabb GDCTM Contest
Level J2 Solutions

1. Amy is in a line of 40 people at the movies. Ahead of her stand 12 people. How many stand behind her?

   (A) 13 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 39

   Solution: **(B)**. Don't forget to count Amy herself. So $12 + 1 + 27 = 40$.

2. Simplify $a + a + a + a + a + a + a + a$.

   (A) $4a$ **(B)** $5a$ **(C)** $6a$ **(D)** $7a$ **(E)** $8a$

   Solution: **(E)** There are 8 a's. Factor out the a to get $8a$.

3. If the least common multiple of a and b is 38, what is the least common multiple of $15a$ and $15b$?

   (A) 38 **(B)** 114 **(C)** 190 **(D)** 570 **(E)** not uniquely determined

   Solution: **(D)**. Since no prime divides both 15 and 38, the effect of multiplying by 15 will just multiply the earlier lcm by 15. Thus $15 \cdot 38 = 570$ is correct.

4. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

   Solution: **(C)** Let the plums be numeraire, that is, assign plum equal to one. Let a pomegranate weigh x and a pineapple weigh y. Solve the system $3x + y = 16$, $4 + x = y$, for example, by substitution. We get $3x + 4 + x = 16$ or $x = 3$. Then $y = 7$. So three pineapples weigh 21 and it takes 7 pomegranates to weigh as much.

5. What is the area of the quadrilateral in the coordinate plane with vertices whose coordinates are (in order): $(0, 0)$, $(7, 1)$, $(4, 4)$, and $(2, 11)$?

   (A) 30 **(B)** 31 **(C)** 31.5 **(D)** 32 **(E)** 33

   Solution: **(A)** This can be done with the shoestring method. Setting up the determinants by going counterclockwise around the quadrilateral gives for the area

   $$\frac{28 + 44 - 4 - 8}{2} = 60/2 = 30$$

6. Ronald has an unlimited number of 5 cent and 7 cent stamps. What is the largest amount of postage (in cents) that he cannot make with these stamps?

   (A) 16 **(B)** 22 **(C)** 23 **(D)** 79 **(E)** 99

   Solution: **(C)** Notice that 24, 25, 26, and 27 cent postage can all be made by Ronald. Once 4 in a row are established like this, all higher amounts of postage is possible, simply by adding the right number of 4 cent stamps. Since 23 cents of postage cannot be made, 23 is then the largest such amount.

7. If r is a solution of the equation $x^2 + 11x + 19 = 0$, what is the value of $(r + 5)(r + 6)$?

(A) -15 **(B)** -11 **(C)** 0 **(D)** 7 **(E)** 11

Solution: **(E)** Observe that $(r+6)(r+5) = r^2+11r+30 = (r^2+11r+19)+11 = 0+11 = 11$.

8. Abigail, Brice, and Carl all start out with some one-dollar bills in their pockets. In particular, Carl starts with 4 one-dollar bills. Brice then gives half of his dollar bills to Abigail and the other half to Carl. Then Abigail gives half of her bills to Brice and the other half to Carl. Finally, Carl gives half of his bills to Abigail and keeps the rest. If now, at the end of these exchanges, Abigail, Brice, and Carl all have 8 one-dollar bills, how many such bills did Abigail have to begin with?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Solution: **(E)** Work backwards. Just before the last exchange Abigail has zero dollar bills (because she has just given away all of her bills), Brice eight, and Carl sixteen. So working back one more exchange, Abigail would have 16 bills, Brice zero, and Carl eight. Since Carl started with four he must gain four on the first exchange, meaning that Abigail also gained four bills on the first exchange, implying she started with 12 bills.

9. If $f(x)$ is a linear function for which $f(8) - f(1) = 11$, then $f(41) - f(6)$ is equal to

(A) 61 **(B)** 55 **(C)** 49 **(D)** 43 **(E)** 37

Solution: **(B)** From the first equation the slope of the line must be 11/7. Therefore $f(41) - f(6) = (11/7)(41 - 6) = (11/7)35 = 11 \cdot 5 = 55$.

10. Let a, b, c, and d be positive real numbers such that $abc = 12$, $bcd = 6$, $acd = 125$, and $abd = 3$. Then $abcd$ equals

(A) 12 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

Solution: **(C)** Multiply all the equations together and take the cube root of both sides.

11. The surface area of a large spherical balloon is doubled. By what factor is the volume of the balloon increased?

(A) 8 **(B)** 4 **(C)** $2\sqrt{2}$ **(D)** $\sqrt[3]{4}$ **(E)** 2

Solution: **(C)** Since the surface area of the sphere is proportional to the square of the radius, then the radius had gone up by a factor of $\sqrt{2}$. As volume is proportional to the cube of the radius, the volume has gone up by a factor of $(\sqrt{2})^3 = 2\sqrt{2}$.

12. Zeke cycles steadily for 36 miles. If he had managed to go 3 mph faster, he would have taken one hour less for the trip. What was Zeke's actual speed in mph during this trip?

(A) 9 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 21

Solution: **(A)** Let x be Zeke's actual speed and $x + 3$ the supposed speed. Then since time equals distance divided by speed we have

$$\frac{36}{x} - \frac{36}{x+3} = 1$$

Solving this equation yields $x = 9$.

13. A set of seven distinct positive integers has a mean of 13. Find the difference between the greatest possible median of these integers and the least possible median of these integers.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: **(D)** The sum of all seven integers is 91. The least possible median of 4 comes about from the set $\{1, 2, 3, 4, 5, 6, 70\}$. The greatest possible median of 19 occurs in the set $\{1, 2, 3, 19, 20, 21, 25\}$, among others. The number 20 cannot be the median as then the smallest possible sum would be $1+2+3+20+21+22+23 = 92$. The answer is $19-4 = 15$.

14. A line L in the coordinate plane has slope -2. Suppose the triangle with vertices given by the origin, the x-intercept of L, and the y-intercept of L has area 9. Then an equation for L could be

(A) $2x + y = 0$ **(B)** $2x + y = 4$ **(C)** $-2x + y = 6$
(D) $2x + y = 3$ **(E)** $2x + y = -6$

Solution: **(E)** Because the slope of the line has absolute value 2, then this right triangle must have legs of length a and $2a$. Thus $(1/2)a(2a) = a^2 = 9$ so that $a = 3$ or $a = -3$. If $a = 3$, then the line does not match any of the equations given. So $a = -3$ does and leads to $2x + y = -6$.

15. In throwing four fair cubical dice, what is the probability of obtaining two distinct doubles?

(A) $\frac{5}{72}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{3}{8}$

Solution: **(A)** There are $\binom{6}{2}$ ways of choosing which 2 numbers will appear as doubles, then $\binom{4}{2}$ ways of choosing which dice get which of those numbers. Multiply theses quantities, then divide by 6^4 to find the probability.

16. If a is a multiple of 14 and b is a multiple of 21, then what is the largest integer that must be a factor of any integer of the form $9a + 8b$?

(A) 84 **(B)** 42 **(C)** 21 **(D)** 14 **(E)** 8

Solution: **(B)** The greatest common factor of 14 and 21 is 7. So $9a + 8b$ is certainly divisible by 7. With 9 being divisible by 3 and 21 being divisible by 3 then $9a + 8b$ is also divisible by 3. And since a is divisible by 2 and 8 is divisible by 2, then $9a + 8b$ is divisible by 2. We can be certain of no further divisibilities, the answer is $7 \cdot 3 \cdot 2$ or 42.

17. Each vertex of a cube is randomly colored red or blue with each color being equally likely. What is the probability that every pair of adjacent vertices have different colors?

(A) 0 **(B)** 1/128 **(C)** 1/64 **(D)** 1/32 **(E)** 1/2

Solution: **(B)** Consider a particular vertex of the cube. It does not matter which one or which color it is. Then the three vertices that are connected by an edge with this chose vertex must have opposite color to it. That happens with probabiltiy $1/2^3$ or $1/8$. These three newly colored vertices force 3 more vertices to be colored the same as the original vertex. This happens with probability 1/8. Finally the last vertex is colored opposite those just determined, with probability 1/2. So the final probability of all this happening is $(1/8)(1/8)(1/2) = 1/128$.

18. One root of $2x^2 + 15x + c$ is four times the other. What is the value of c?

(A) 36 **(B)** 9 **(C)** 18 **(D)** 3/2 **(E)** -3/2

Solution: **(C)** By Viete's Theorem, if the roots are called r and $4r$, than $r+4r = -15/2 = 5r$ and $r \cdot 4r = c/2$. From the former, $r = -3/2$. From the latter $c = 8r^2 = 8(9/4) = 18$.

19. The product of three distinct positive integers is 210. What is the maximum possible sum of these three integers?

(A) 18 **(B)** 38 **(C)** 74 **(D)** 108 **(E)** 212

Solution: **(D)** The prime factorization of 210 is $2 \cdot 3 \cdot 5 \cdot 7$. The integer 1 can be used just once, so the largest product with two distinct factors greater that one will occur with 2 and 105. So the largest such sum is $1 + 2 + 105 = 108$.

20. The value of the expression

$$(1 - (2 - (3 - (4 - (5 - (\cdots - (n)))) \cdots)$$

for n a positive even integer is equal to

(A) $-n$ **(B)** $-n/2$ **(C)** 0 **(D)** $n/2$ **(E)** $n - 3$

Solution: **(B)** Removing parentheses when n is even results in $1-2+3-4+5-6+\cdots+(n-1)-n$. Grouping in pairs as $(1-2)+(3-4)+\cdots(n-1-n) = -1+(-1)+(-1)+\cdots+(-1)$ shows that there are $n/2$ negative ones being added. Thus they add to $-n/2$.

21. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1 inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

Solutions: **(A)** Let (i, j) stand for the small square in the ith row and jth column. There are 4! or 24 ways to color the first row. There are 3 ways to select the color for $(2, 1)$. Withoug loss of generality we can assume that the color that went in $(2, 1)$ is the same as the color that went in $(1, 4)$. There arise two cases depending on the color for $(2, 4)$.

Case I. Color $(2, 4)$ the same as $(1, 1)$. Then the colors for the remaining squares of row 2 are determined. For coloring $(3, 1)$ and $(3, 2)$ there are 2 ways. This choice forces the color choices in the remaining squares of column 4. Likewise there are two choices for the last 2 squares of column 2, which force the choices for the last two squares of column three. Counting the previous choices Case I gives $4! \cdot 3 \cdot 2 \cdot 2$ or 288 choices.

Case II. Color $(2, 4)$ the same as either $(1, 2)$ or $(2, 2)$. This gives 2 choices and forces the colors for the remaining squares of row 2. For the last two squares of the first column there are 2 choices. One can check that this last choice forces the remaining 6 squares. So Case II gives altogether 288 choices as well, though it does not seem obvious why this should be so. I would be interested in an argument of why it must be so if it must be so. (What would happen if it where a 5 by 5 square with 5 colors? Is it also the square of 5!??)

Adding our two disjoint cases together we obtain $288 + 288$ or $576 = 24^2$.

22. Two ferries start at the same instant from opposite banks of a river. They travel directly across the river. Each boat keeps its own constant speed, though one boat is faster than the other. In this first trip across they pass at a point 720 yards from the nearer bank. When reaching the opposite shore each boat remains exactly 10 minutes in its dock before heading back the other way. On this trip back the boats meet 400 yards from the other shore. How wide is the river (in yards)?

(A) 1040 **(B)** 1120 **(C)** 1520 **(D)** 1600 **(E)** 1760

Solution: **(E)** Let x be the distance the faster ferrie travels from its first departure until it meets the slower ferry coming the other way. At this point both ferries are 720 yards from the nearer shore, which is where the slower ferry must have departed. As the speeds of the ferries remain constant the ratio of the distances they travel in an equal time must remain the same. By drawing a careful diagram you can determine that from their first meeting until their second the faster ferry has travelled at total of $x + 320 + 720$ while the slower has travelled at total of $x + 400$. Thus

$$\frac{x}{720} = \frac{x + 320 + 720}{x + 400}$$

Thus $x^2 - 320x - 720(720 + 320) = (x - (720 + 320))(x + 720) = 0$. Since $x > 0$ we must have $x = 720 + 320 = 1040$. The width of the river is $x + 720 = 1040 + 720 = 1760$. There is a simpler way to do this problem. Try to find it! This problem was created in the nineteenth century by Sam Lloyd, the great American problem poser.

23. In how many ways can five distinct books be arranged in a bookcase with 3 shelves, each shelf capable of holding all five books?

(A) 19 **(B)** 120 **(C)** 360 **(D)** 840 **(E)** 2520

Solution: **(E)** First we count the number of ways to assign how many books each shelf gets. To the five books, add two dividers to distinguish the shelves, giving, we imagine, 7 blanks in a row. By choosing two of the blanks to be dividers (so the remaining blanks become books) we determine how many books go on each shelf. This can be done in $\binom{7}{2}$ or 21 ways. Next we can assign particular books to each of the five spots just determined in 5! or 120 ways. The total count of ways to arrange the books in the bookcase becomes $21 \cdot 120$ or 2520.

24. The probability that Gerald wins any given game of HORSE is 3/5. Next Saturday, Gerald will play exactly five games of HORSE. What is the probability that he will win exactly three of them?

(A) $\frac{108}{3125}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{216}{625}$ **(D)** $\frac{9}{25}$ **(E)** 1

Solution: **(C)** There are $\binom{5}{3}$ ways to choose which of the three games Gerald wins. For each of these choices, the probability it happens that way comes to $(3/5)^3(2/5)^2$. The probability Gerald does win exactly three games is $\binom{5}{3}(3/5)^3(2/5)^2 = 216/625$.

25. Seven stacks are made each consisting of seven half-dollar coins. One entire stack is made of counterfeit coins. All other stacks have true half-dollars. You know the weight of true half-dollars in grams. And each counterfeit half-dollar weighs exactly one gram more than the true coin. You can weigh the coins or any subset of them on a digital scale (similar to a regular bathroom scale) which outputs in grams. What is the minimum number of weighings needed to determine which stack is the counterfeit one?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 7

Solution: **(A)** Surprisingly, this can be done in one weighing! Number the stacks from 1 to 7. Take one coin from stack #1 , two coins from stack #2, etc, all the way to all seven coins from stack # 7. Put all these coins on the scale. Their weight will be k grams over the weight of that many genuine coins. This can only occur if stack # k is the counterfeit one.