Fall 2011 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. The sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

is equal to

(A) 1/10 **(B)** 1/24 **(C)** 1/6 **(D)** 25/6 **(E)** 25/12

Solution: (E) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=3/2+7/12=(18+7)/12=25/12$

2. A certain number is doubled. The result is then increased by nine. This result is decreased by 3. If this last number is 28, what was the original number?

(A) -4 **(B)** 0 **(C)** 7 **(D)** 11 **(E)** 28

Solution: (D) Work backwards: 28 to 31 to 22 to 11.

3. A train travels 1 mile in 1 minute and 20 seconds. At this speed, how many miles will the train travel in 112 minutes?

(A) 84 **(B)** 86 **(C)** 88 **(D)** 90 **(E)** 96

Solution: (A) $\frac{x}{112}=\frac{1}{4/3}$, so $x=112\cdot(3/4)=84$.

4. What is the number of square inches in a rectangle which measures $1\frac{1}{4}$ feet by $1\frac{1}{6}$ yards?

(A) 35/24 **(B)** 35/2 **(C)** 70 **(D)** 120 **(E)** 630

Solution: (E) $\frac{5}{4}\cdot 12\cdot\frac{7}{6}\cdot 36=15\cdot 42=630$.

5. Two sweaters, a pair of wool socks, and a coat cost \$180. One sweater and the coat cost \$130. How much does one sweater and a pair of wool socks cost?

(A) \$30 **(B)** \$40 **(C)** \$50 **(D)** \$60 **(E)** \$70

Solution: (C) Subtract the orders.

6. Five siblings, each a different age, split a gift of \$200 in such a way that each child other than the youngest, gets ten dollars more than the next younger sibling. The youngest, of course, gets ten dollars less than the next to youngest. How much does the middle child receive?

(A) \$25 **(B)** \$30 **(C)** \$35 **(D)** \$40 **(E)** \$45

Solution: (D) The middle child gets the average gift: $200/5=40$.

7. A given cone's dimensions are modified as described in the responses below. Which response does **not** change the volume?

   (A) double the height and halve the radius
   (B) halve the height and double the radius
   (C) quadruple the height and halve the radius
   (D) halve the height and quadruple the radius
   (E) quadruple the height and halve the radius twice

   Solution: (C) The volume of a cone is $(1/3)\pi r^2 h$. Replacing h by $4h$ and r by $r/2$ leaves the volume invariant.

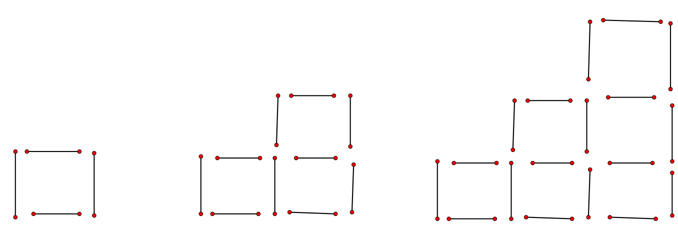
8. The sum of the first n positive integers is 210. What is the average of these first n positive integers?

   (A) 9 **(B)** 9.5 **(C)** 10 **(D)** 10.5 **(E)** 21

   Solution: (D) The sum of the first 20 positive integers is 210. So the average is $210/20 = 21/2 = 10.5$.

9. In a sequence of of matchstick diagrams the next diagram adds one more column of blocks built one block higher and attached to the right of the previous diagram. Shown are the first three diagrams in this sequence. What is the number of matchsticks in the 7th diagram of this sequence?

   (A) 70 **(B)** 72 **(C)** 74 **(D)** 76 **(E)** 78



   Solution: (A) The number of matchsticks is $4 + 6 + 8 + 10 + 12 + 14 + 16 = 70$.

10. Amanda, Brice, and Carl all start working at SellMore with the same salary on the same day. They receive the following percent raises in their salary at the end of each of the first two years in order:

    Amanda: 5%, 3%

    Brice: 3%, 5%

    Carl: 4%, 4%

    Which of them earns the most total over their first three years at SellMore?

    (A) Amanda **(B)** Brice **(C)** Carl
    (D) Amanda and Brice tied for the most **(E)** All three tied for the most

Solution: (A) Assume they all earn 1 the first year. So we just worry about the sum of the salaries over the last two years.
For Amanda: 1(1.05)+1(1.05)(1.03)=(1.05)(2.03)=2.1315.
For Brice: 1(1.03)+1(1.03)(1.05)=(1.03)(2.05)=2.1115.
For Carl: 1(1.04)+1)(1.04)(1.04)=(1.04)(2.04)=2.1216.
So Amanda has the greatest total salary over the past three years.

11. From a regular deck of 52 cards three cards are dealt to you. What is the probability all three are red cards? Recall the red suites are hearts and diamonds.

(A) $2/17$ **(B)** $1/8$ **(C)** $2/15$ **(D)** $1/7$ **(E)** $2/13$

Solution: (A) $\frac{26}{52} \cdot \frac{25}{51} \cdot \frac{24}{50} = 2/17.$

12. The value of
$$\frac{1^3 + 2^3 + 3^3 + 4^3 + \cdots + 17^3}{4^3 + 8^3 + 12^3 + 16^3 + \cdots + 68^3}$$
is equal to

(A) $1/4$ **(B)** $1/16$ **(C)** $1/64$ **(D)** $1/68$ **(E)** $1/192$

Solution: (C) $4^3 + 8^3 + 12^3 + 16^3 + \cdots + 68^3 = 4^3(1^3 + 2^3 + 3^3 + 4^3 + \cdots + 17^3)$

13. In how many ways can 10 be written as a sum of one or more positive integers if order does not matter and no integer can be repeated in a given sum? Thus, for instance, $4 + 6$ is considered the same as $6 + 4$, and $5 + 5$ is not allowed.

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Solution: (E) The ten ways are: 10, 9+1, 8+2, 7+3, 7+2+1, 6+4, 6+3+1, 5+4+1, 5+3+2, 4+3+2+1.

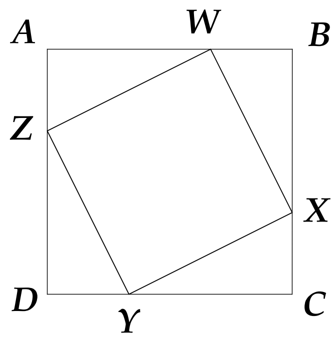
14. In square $ABCD$, square $WXYZ$ is inscribed in such a way that W is two-thirds of the way from A to B, X is two-thirds of the way from B to C, Y is two-thirds of the way from C to D, and Z is two-thirds of the way from D to A. If the area of $WXYZ$ is 100, what is the area of $ABCD$?

(A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 200

Solution: (D) Let $WB = x$, so $BX = 2x$ and $x^2 + (2x)^2 = 100 = 5x^2$, so $9x^2 = 180$.

15. If a raindrop has a volume of 10 cubic millimeters, a certain school yard has dimensions 50 meters by 40 meters, and this yard receives 5 centimeters of rain, the number of raindrops that fell on the yard is

(A) 10^9 **(B)** 10^{10} **(C)** 10^{11} **(D)** 10^{12} **(E)** 10^{13}



Solution: (B) Find total volume of rain in cubic millimeters and divide by 10, so that

$$\frac{50 \cdot 10^3 \cdot 40 \cdot 10^3 \cdot 50}{10} = 5 \cdot 5 \cdot 4 \cdot 10^1 \cdot 10^3 \cdot 10^1 \cdot 10^3 = 100 \cdot 10^8 = 10^{10}$$

16. If $n^3 = 18 \cdot 96$, then n^2 is equal to

(A) 36 **(B)** 81 **(C)** 121 **(D)** 144 **(E)** 196

Solution: (D) $18 \cdot 96 = 27 \cdot 64 = 12^3$, so $n = 12$ and $n^2 = 144$.

17. Seven consecutive integers are written on a whiteboard. When one of them is erased, the sum of the remaining six integers is 857. What is the sum of the original seven integers?

(A) 1001 **(B)** 1011 **(C)** 1111 **(D)** 1112 **(E)** 1123

Solution: (A) 857 divided by 6 is just a bit less than 143 so we try with 143 in the middle position. The sum $140 + 141 + 142 + 143 + 144 + 145 + 146 = 1001$ and $1001 - 857 = 144$, one of the seven original numbers. It works! Others don't! For example, $141 + 142 + 143 + 144 + 145 + 146 + 147 = 1008$ and $1008 - 857 = 151$, but 151 is not in that list!

18. How many factors of $51^5 \cdot 71^7 \cdot 91^9$ are perfect squares?

(A) 1 **(B)** 60 **(C)** 180 **(D)** 192 **(E)** 900

Solution: (E) Must use prime factorization: $3^5 \cdot 7^9 \cdot 13^9 \cdot 17^5 \cdot 71^7$. For a perfect square, the exponents must be even. So there are $3 \cdot 5 \cdot 5 \cdot 3 \cdot 4 = 900$ such choices. For example out of the 3^5 one can take 3^0, 3^2, or 3^4 giving 3 choices, etc...

19. Hezy and Zeke have a 6 hour drive to get to their grandparents house for Thanksgiving. Each will drive on their turn(s), if they have a turn, a positive whole number of hours. They can switch drivers or not as they wish, so long as they follow the rule of each driver driving a whole number of hours on their turn(s). They could even not switch at all. If Hezy starts the trip, in how many different ways of sharing (or not!) the driving, can they get to their grandparents?

(A) 24 **(B)** 30 **(C)** 32 **(D)** 64 **(E)** 120

Solution: (C) For each of the remaining 5 hours either Hezy or Zeke are driving. So 2^5 ways to decide who is driving during each of those hours.

20. In how many ways can a group of ten students be split into two groups of five each?

(A) 50 **(B)** 63 **(C)** 126 **(D)** 252 **(E)** $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

Solution: (C) Choose 5 from the 10 but then divide by 2. To see why imagine five are boys and five are girls. If you choose all five boys you get the same split as when choosing all five girls. So $\binom{10}{5}/2 = 126$ ways.

21. The product of the repeating decimals $0.\overline{3}$ and $0.\overline{12}$ is

(A) $0.\overline{03}$ **(B)** $0.\overline{04}$ **(C)** $0.\overline{36}$
(D) $0.\overline{6}$ **(E)** not repeating

Solution: (B) $0.\overline{3} \cdot 0.\overline{12} = (1/3)(12/99) = 4/99 = 0.\overline{04}$

22. Blindfolded, Sue rolls two standard cubical dice. Her friend tells her that the sum of the two numbers rolled is less than six. What is the probability that Sue rolled snake-eyes, that is, two ones?

(A) 1/36 **(B)** 1/18 **(C)** 1/12 **(D)** 1/11 **(E)** 1/10

Solution: (E) The sample spaces shrinks to just ten events, one of which is snake-eyes. The ten events are: $(4,1),(3,2),(2,3),(1,4),(3,1),(2,2),(1,3),(2,1),(1,2), and (1,1)$.

23. Which of the following *cannot* be the number of zeros in which $n!$ ends?

(A) 148 **(B)** 150 **(C)** 152 **(D)** 154 **(E)** 156

Solution: (D) Note that 625! ends in $125 + 25 + 5 + 1 = 156$ zeros, while 624! ends in 152 zeros since $625 = 5^4$.

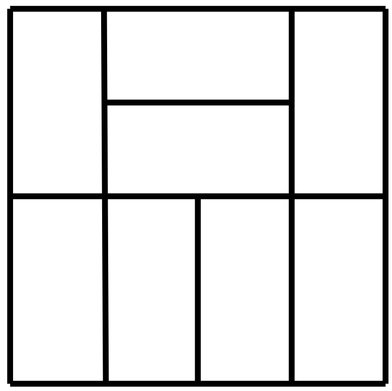
24. What is the smallest 4 digit prime number?

(A) 1001 **(B)** 1003 **(C)** 1005 **(D)** 1007 **(E)** 1009

Solution: (E) Note that 1001 is divisible by 7, 1003 divisible by 17, 1005 divisible by 5, and 1007 by 19.

25. In how many ways can a 4×4 nailed down board be tiled by eight 1×2 dominoes? One way to tile the board is shown below.

(A) 16 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 49



Solution: (C) Split into cases. If no domino crosses the horizontal midline of the square there are 5^2 or 25 ways to tile, since there are 5 ways to tile a 2 by 4 rectangle. An even number of dominoes must cross the midline. If exactly two dominoes cross the midline there are 10 ways. These 10 further split as: just one way if the two that cross are not next to each other (they must be at opposite ends of the midline); four ways if they are next to each other far left (because of the five ways to do a 4 by 2 minus the one way that would lead to 4 dominoes crossing the midline); four ways if they are next to each other far right; and one way if the two occupy the center. Finally if four dominoes cross the midline, there is only one way. So there are $25 + 10 + 1 = 36$ ways to tile the 4 by 4 board with 1 by 2 dominoes.

Fall 2011 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

1. A certain number is doubled. The result is then increased by nine. This result is decreased by 3. If this last number is 28, what was the original number?

   (A) -4 **(B)** 0 **(C)** 7 **(D)** 11 **(E)** 28

   Solution: (D) Work backwards: 28 to 31 to 22 to 11.

2. Two sweaters, a pair of wool socks, and a coat cost \$180. One sweater and the coat cost \$ 130. How much does one sweater and a pair of wool socks cost?

   (A) \$30 **(B)** \$40 **(C)** \$50 **(D)** \$60 **(E)** \$70

   Solution: (C) Subtract the orders.

3. The expression
$$3\big((a+3b)4+2(5b+a)\big)$$
   is equivalent to the expression

   (A) $18a+22b$ **(B)** $15a+22b$ **(C)** $42a+42b$
   (D) $15a+66b$ **(E)** $18a+66b$

   Solution: (E) Distribute and collect: $3(4a+12b+10b+2a) = 3(6a+22b) = 18a+66b$.

4. How many subsets of $\{a,b,c,d,e\}$ have an odd number of elements?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

   Solution: (E) Add every other element of the fifth row of Pascal's Triangle: $5+10+1=16$.

5. From a regular deck of 52 cards three cards are dealt to you. What is the probability all three are red cards? Recall the red suites are hearts and diamonds.

   (A) 2/17 **(B)** 1/8 **(C)** 2/15 **(D)** 1/7 **(E)** 2/13

   Solution: (A) $\frac{26}{52}\cdot\frac{25}{51}\cdot\frac{24}{50} = 2/17.$

6. Xenia is three years older than Zyler. Eight years ago Zyler was half the age of Xenia. How many years from now will Xenia be 8/7 ths the age of Zyler?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

   Solution: (C) Let $x =$ the age of Xenia now and let $z =$ the age of Zyler now. Then $x = z + 3$ and $z - 8 = (x - 8)/2$. Thus $z - 8 = (z - 5)/2$ or $2z - 16 = z - 5$ or $z = 11$. So $x = 14$. Let $y =$ number of years in future so that $14 + y = (8/7)(11 + y)$. Then $98 + 7y = 88 + 8y$ or $y = 10$.

7. On the real number line, let point A have coordinate a and point B have coordinate b. What is the coordinate of the point between A and B which is four times closer to B than it is to A?

   (A) $\dfrac{4a+b}{5}$ **(B)** $\dfrac{3a+b}{4}$ **(C)** $\dfrac{a+4b}{5}$ **(D)** $\dfrac{a+3b}{4}$ **(E)** $\dfrac{a+b}{2}$

   Solution: (C) Weight B four times as much as A and average.

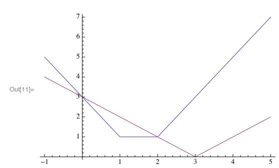
8. The number of solutions of the equation

$$|x-1| + |x-2| = |x-3|$$

   is equal to

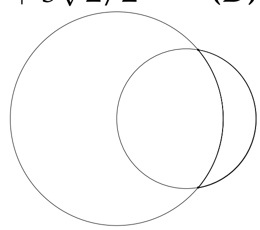
   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: (C) Two solutions are $x = 0$ and $x = 2$. Use cases such as $x < 1$, $1 \leq x < 2$, etc, to check there are no other solutions. Or else graph the two sides and see where they intersect:



9. The distance between the two centers of the two circles is 3, the center of the larger circle lies on the smaller circle, and the points of intersection of the two circles lie on the same diameter of the smaller circle. Find the area of that part of the smaller circle that lies outside of the larger circle.

   (A) 9 **(B)** $\pi/4$ **(C)** $9\pi/8 + 3\sqrt{2}/2$ **(D)** $3\pi - 3$ **(E)** 10



   Solution: (A) The radius of the smaller circle is 3 and the radius of the larger circle is $3\sqrt{2}$. The desired area is the difference of the area of a semicircle of the smaller circle and a region that is itself the difference between a quarter circle of the larger circle and a right isosceles triangle with leg equal to the radius of the larger circle. Thus we obtain our answer is

$$9\pi/2 - (18\pi/4 - (1/2)18) = 18/2 = 9$$

10. If $a < b < c < d < e$ which of the following must be true?

   (A) $ab < cd$
   (B) $c - a < e - c$
   (C) $a^2 < e^2$
   (D) $ad + bc < ac + bd$
   (E) $b + d < 2c$

   Solution: (D) (D) is equivalent to $ac - ad + bd - bc = (b-a)(d-c) > 0$.

11. If a raindrop has a volume of 10 cubic millimeters, a certain school yard has dimensions 50 meters by 40 meters, and this yard receives 5 centimeters of rain, the number of raindrops that fell on the yard is

   (A) 10^9 **(B)** 10^{10} **(C)** 10^{11} **(D)** 10^{12} **(E)** 10^{13}

   Solution: (B) Find total volume of rain in cubic millimeters and divide by 10, so that

$$\frac{50 \cdot 10^3 \cdot 40 \cdot 10^3 \cdot 50}{10} = 5 \cdot 5 \cdot 4 \cdot 10^1 \cdot 10^3 \cdot 10^1 \cdot 10^3 = 100 \cdot 10^8 = 10^{10}$$

12. Find the sum of all the factors of 280.

   (A) 440 **(B)** 540 **(C)** 600 **(D)** 640 **(E)** 720

   Solution: (E) $280 = 2^3 \cdot 5 \cdot 7$ so the sum of the factors equals $(1+2+4+8)(1+5)(1+7) = 15 \cdot 6 \cdot 8 = 720$

13. The expression

$$a-(b+(c-(d+(e-f))))$$

is equivalent to

(A) $a-b-c+d+e-f$
(B) $a-b-c+d-e-f$
(C) $a-b-c+d+e+f$
(D) $a-b+c-d+e-f$
(E) $a-b+c-d+e-f$

Solution: (A) $a-(b+(c-(d+(e-f)))) = a-(b+(c-d-e+f)) = a-b-c+d+e-f$

14. The product of the repeating decimals $0.\overline{3}$ and $0.\overline{12}$ is

(A) $0.\overline{03}$ **(B)** $0.\overline{04}$ **(C)** $0.\overline{36}$
(D) $0.\overline{6}$ **(E)** not repeating

Solution: (B) $0.\overline{3}\cdot 0.\overline{12} = (1/3)(12/99) = 4/99 = 0.\overline{04}$

15. Amanda, Brice, and Carl all start working at SellMore with the same salary on the same day. They receive the following percent raises in their salary at the end of each of the first two years in order:

Amanda: 5%, 3%

Brice: 3%, 5%

Carl: 4%, 4%

Which of them earns the most total over their first three years at SellMore?

(A) Amanda **(B)** Brice **(C)** Carl
(D) Amanda and Brice tied for the most **(E)** All three tied for the most

Solution: (A) Though the total sum of the percent raises is the same for all three employees, it is best to get the largest raise first as a simple calculation shows.

16. What is the smallest 4 digit prime number?

(A) 1001 **(B)** 1003 **(C)** 1005 **(D)** 1007 **(E)** 1009

Solution: (E) Note that 1001 is divisible by 7, 1003 divisible by 17, 1005 divisible by 5, and 1007 by 19.

17. If p people consume m pounds of mashed potato in h hours, then the pounds of mashed potato consumed by m people in p hours equals:

(A) mph **(B)** $\dfrac{m}{ph}$ **(C)** $\dfrac{m^2}{ph}$ **(D)** $\dfrac{m^2}{h}$ **(E)** $\dfrac{p^2}{m}$

Solution: (D) The pounds z consumed is jointly proportional to the number of people x and the number of hours y. From the given, $m = kph$ so that the constant of proportionality k equals $m/(ph)$. Thus $z = (m/(ph))mp = m^2/h$.

18. If a positive integer n has exactly 12 factors, what is the difference between the greatest and least number of factors that n^2 could have?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 27

Solution: (A) Let p, q, and r denote primes.

If $n = p^{11}$ then $n^2 = p^{22}$ has 23 factors.

If $n = p^5 q$ then $n^2 = p^{10} q^2$ has 33 factors.

If $n = p^3 q^2$ then $n^2 = p^6 q^4$ has 35 factors.

If $n = p^2 qr$ then $n^2 = p^4 q^2 r^2$ has 45 factors.

So 45-23=22.

19. Hezy and Zeke have a 6 hour drive to get to their grandparents house for Thanksgiving. Each will drive on their turn(s), if they have a turn, a positive whole number of hours. They can switch drivers or not as they wish, so long as they follow the rule of each driver driving a whole number of hours on their turn(s). They could even not switch at all. If Hezy starts the trip, in how many different ways of sharing (or not!) the driving, can they get to their grandparents?

(A) 12 **(B)** 24 **(C)** 30 **(D)** 32 **(E)** 64

Solution: (D) For each of the remaining 5 hours either Hezy or Zeke are driving. So 2^5 ways to decide who is driving during each of those hours.

20. Blindfolded, Sue rolls two standard cubical dice. Her friend tells her that the sum of the two numbers rolled is less than six. What is the probability that Sue rolled snake-eyes, that is, two ones?

(A) $1/36$ **(B)** $1/18$ **(C)** $1/12$ **(D)** $1/11$ **(E)** $1/10$

Solution: (E) The sample spaces shrinks to just ten events, one of which is snake-eyes. The ten events are: $(4,1),(3,2),(2,3),(1,4),(3,1),(2,2),(1,3),(2,1),(1,2), and (1,1)$.

21. How many of the numbers in this set below are irrational?

$$\{\sqrt{1.00}, \sqrt{1.01}, \sqrt{1.02}, \sqrt{1.03}, \cdots, \sqrt{3.98}, \sqrt{3.99}\}$$

(A) 299 **(B)** 294 **(C)** 290 **(D)** 286 **(E)** 150

Solution: (C) Of the 300 numbers in the list ten are rational, having the form $\sqrt{n^2/100}$ for $n = 10, 11, 12, \cdots, 19$.

22. Find the sum

$$1 \cdot 25 + 2 \cdot 24 + 3 \cdot 23 + 4 \cdot 22 + \cdots + 24 \cdot 2 + 25 \cdot 1$$

(A) 2500 **(B)** 2725 **(C)** 2800 **(D)** 2825 **(E)** 2925

Solution: (E) The sum is $\binom{27}{3}$. Imagine the sum as the number of blocks in a structure with, working left to right, one column 25 blocks high, then two columns each 24 blocks high, etc.. Now read this by rows. The top row has 1 block. The second has 1+2 blocks. The third has 1+2+3 blocks,etc.. till the bottom row has $1 + 2 + 3 + 4 + \cdots + 25$ blocks. So this sum overall is the sum of the first 25 triangular numbers. Looking at the triangular numbers as a diagonal of Pascal's Triangle, and using the hook identity we see the sum is $\binom{27}{3} = 2925$. Can you find a purely combinatorial solution?

23. How many factors of $51^5 \cdot 71^7 \cdot 91^9$ are perfect squares?

(A) 1 **(B)** 60 **(C)** 180 **(D)** 192 **(E)** 900

Solution: (E) Must use prime factorization: $3^5 \cdot 7^9 \cdot 13^9 \cdot 17^5 \cdot 71^7$. For a perfect square, the exponents must be even. So there are $3 \cdot 5 \cdot 5 \cdot 3 \cdot 4 = 900$ such choices. For example out of the 3^5 one can take 3^0, 3^2, or 3^4 giving 3 choices, etc...

24. If $a = \dfrac{1110}{1111}$, $b = \dfrac{2221}{2223}$, and $c = \dfrac{3331}{3334}$ which of the following is true?

(A) $a > b > c$ **(B)** $b > a > c$ **(C)** $c > a > b$ **(D)** $c > b > a$ **(E)** $b > c > a$

Solution: (E) Note that $\frac{1}{a} - 1 > \frac{1}{c} - 1 > \frac{1}{b} - 1$ so $b > c > a$.

25. In how many ways can the letters in the word *monsoon* be arranged so that the second n occurs before the third o?

(A) 210 **(B)** 216 **(C)** 252 **(D)** 256 **(E)** 260

Solution: (C) It is somewhat easier to count the complement. So we count the number of ways the second n occurs after the third o.

Case 1. The third o occurs at the third letter. $4 \cdot 3$ ways

Case 2. The third o occurs at the fourth letter. $3 \cdot 4 \cdot 3$ ways

Case 3. The third o occurs at the fifth letter. $\binom{4}{2} \cdot 4 \cdot 3 - \binom{4}{2} \cdot 1 \cdot 2!$ ways.

Case 4. The third o occurs at the sixth letter. $\binom{5}{2} \cdot 3!$ ways

As there are $7!/(3!2!) = 420$ ways to arrange *monsoon* altogether, our answer is $420 - 12 - 36 - 72 + 12 - 60 = 252$.

Fall 2011 McNabb GDCTM Contest
Geometry Solutions

NO Calculators Allowed

1. Two sweaters, a pair of wool socks, and a coat cost \$180. One sweater and the coat cost \$ 130. How much does one sweater and a pair of wool socks cost?

   (A) \$30 **(B)** \$40 **(C)** \$50 **(D)** \$60 **(E)** \$70

   Solution: (C) Subtract the orders.

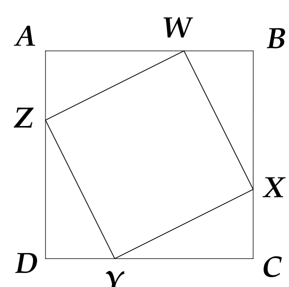
2. The expression
$$3\big((a+3b)4+2(5b+a)\big)$$
   is equivalent to the expression

   (A) $18a+22b$ **(B)** $15a+22b$ **(C)** $42a+42b$
   (D) $15a+66b$ **(E)** $18a+66b$

   Solution: (E) Distribute and collect: $3(4a+12b+10b+2a) = 3(6a+22b) = 18a+66b$.

3. In square $ABCD$, square $WXYZ$ is inscribed in such a way that W is two-thirds of the way from A to B, X is two-thirds of the way from B to C, Y is two-thirds of the way from C to D, and Z is two-thirds of the way from D to A. If the area of $WXYZ$ is 100, what is the area of $ABCD$?

   (A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 200



   Solution: (D) Let $WB = x$, so $BX = 2x$ and $x^2 + (2x)^2 = 100 = 5x^2$, so $9x^2 = 180$.

4. From a regular deck of 52 cards three cards are dealt to you. What is the probability all three are red cards? Recall the red suites are hearts and diamonds.

   (A) 2/17 **(B)** 1/8 **(C)** 2/15 **(D)** 1/7 **(E)** 2/13

   Solution: (A) $\frac{26}{52}\cdot\frac{25}{51}\cdot\frac{24}{50} = 2/17$.

5. If $f(3x+1) = \dfrac{2}{x+4}$, then $f(x+3) =$

   (A) $\dfrac{6}{x+14}$ **(B)** $\dfrac{6}{3x+14}$ **(C)** $\dfrac{6}{x+11}$ **(D)** $\dfrac{6}{x+17}$ **(E)** $\dfrac{2}{x+14}$

   Solution: (A) Replace x by $x-1/3$ in the first given equation to obtain $f(3x) = 2/(x+11/3) = 6/(3x+11)$. This gives $f(x) = 6/(x+11)$ so that $f(x+3) = 6/(x+14)$.

6. A given cone's dimensions are modified as described in the responses below. Which response does **not** change the volume?

(A) double the height and halve the radius
(B) halve the height and double the radius
(C) quadruple the height and halve the radius
(D) halve the height and quadruple the radius
(E) quadruple the height and halve the radius twice

Solution: (C) The volume of a cone is $(1/3)\pi r^2 h$. Replacing h by $4h$ and r by $r/2$ leaves the volume invariant.

7. In how many ways can 10 be written as a sum of one or more positive integers if order does not matter and no integer can be repeated in a given sum? Thus, for instance, $4+6$ is considered the same as $6+4$, and $5+5$ is not allowed.

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Solution: (E) The ten ways are: 10, 9+1, 8+2, 7+3, 7+2+1, 6+4, 6+3+1, 5+4+1, 5+3+2, 4+3+2+1.

8. Find the area enclosed by the graph of

$$|2y-1| + |2y+1| + 2|x| = 4$$

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4

Solution: (C) The equation graphs as a hexagon with vertices at

$$(0,1), (1,1/2), (1,-1/2), (0,-1), (-1,-1/2), (-1,1/2)$$

The y axis splits this hexagon into two congruent hexagons. So the area of the hexagon equals $2 \cdot (1/2) \cdot (2+1) \cdot 1 = 3$.

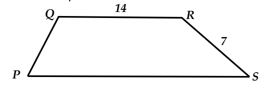
9. An ordered pair (m, n) of positive integers is called a *three-pair* if the interior angle of a regular polygon with m sides is three times the exterior angle of a regular polygon with n sides. How many *three-pairs* exist?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** more than 6

Solution: (D) From $\dfrac{180(m-2)}{m} = 3 \cdot \dfrac{360}{n}$ we solve for n, yielding $n = \dfrac{6m}{m-2}$. So $m = 3, 4, 5, 6, 8, 14$ are the only positive integer values of m for which n is a positive integer greater than two. This follows because for $m > 14$, $6 < \dfrac{6m}{m-2} < 7$.

10. In trapezoid $PQRS$ as shown with $PS \parallel QR$, $QR = 14$, $RS = 7$, and $\angle R = 2\angle P$, the length of PS is

(A) 15 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 27
Solution: (C) Locate point T on $\overline{PS}$ so that $RT \parallel QP$. Then $PQRT$ is a parallelogram, so that $PT = 14$. Also, $\triangle SRT$ is isosceles with $RS = TS = 7$. Thus $PT + TS = PS = 14 + 7 = 21$.



11. If p people consume m pounds of mashed potato in h hours, then the pounds of mashed potato consumed by m people in p hours equals:

(A) mph **(B)** $\frac{m}{ph}$ **(C)** $\frac{m^2}{ph}$ **(D)** $\frac{m^2}{h}$ **(E)** $\frac{p^2}{m}$

Solution: (D) The pounds z consumed is jointly proportional to the number of people x and the number of hours y. From the given, $m = kph$ so that the constant of proportionality k equals $m/(ph)$. Thus $z = (m/(ph))mp = m^2/h$.

12. Find the sum

$$1 \cdot 25 + 2 \cdot 24 + 3 \cdot 23 + 4 \cdot 22 + \cdots + 24 \cdot 2 + 25 \cdot 1$$

(A) 2500 **(B)** 2725 **(C)** 2800 **(D)** 2825 **(E)** 2925

Solution: (E) The sum is $\binom{27}{3}$. Imagine the sum as the number of blocks in a structure with, working left to right, one column 25 blocks high, then two columns each 24 blocks high, etc.. Now read this by rows. The top row has 1 block. The second has 1+2 blocks. The third has 1+2+3 blocks,etc.. till the bottom row has $1 + 2 + 3 + 4 + \cdots + 25$ blocks. So this sum overall is the sum of the first 25 triangular numbers. Looking at the triangular numbers as a diagonal of Pascal's Triangle, and using the hook identity we see the sum is $\binom{27}{3} = 2925$. Can you find a purely combinatorial solution?

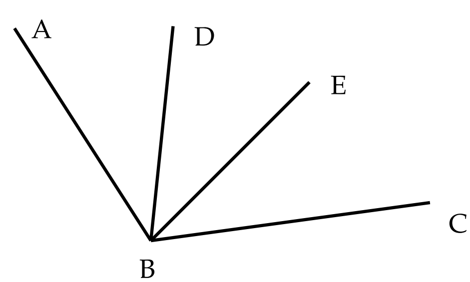
13. How many of the numbers in this set below are irrational?

$$\{\sqrt{1.00}, \sqrt{1.01}, \sqrt{1.02}, \sqrt{1.03}, \cdots, \sqrt{3.98}, \sqrt{3.99}\}$$

(A) 299 **(B)** 294 **(C)** 290 **(D)** 286 **(E)** 150

Solution: (C) Of the 300 numbers in the list ten are rational, having the form $\sqrt{n^2/100}$ for $n = 10, 11, 12, \cdots, 19$.

14. $\angle ABC$ is trisected by rays $\overrightarrow{BD}$ and $\overrightarrow{BE}$ as shown. If the degree measure of $\angle DBC$ equals $3x - 5$ and that of $\angle ABC$ equals $5x - 22$, find the value of x.



(A) 17 **(B)** 19 **(C)** 23 **(D)** 29 **(E)** 31

Solution: (D) Then $(3/2)(3x - 5) = 5x - 22$ or $9x - 15 = 10x - 44$ or $x = 44 - 15 = 29$.

15. How many factors of $51^5 \cdot 71^7 \cdot 91^9$ are perfect squares?

(A) 1 **(B)** 60 **(C)** 180 **(D)** 192 **(E)** 900

Solution: (E) Must use prime factorization: $3^5 \cdot 7^9 \cdot 13^9 \cdot 17^5 \cdot 71^7$. For a perfect square, the exponents must be even. So there are $3 \cdot 5 \cdot 5 \cdot 3 \cdot 4 = 900$ such choices. For example out of the 3^5 one can take 3^0, 3^2, or 3^4 giving 3 choices, etc...

16. In acute triangle ABC, the intersection of its three altitudes, called the *orthocenter*, is labeled P. Given that $AP = 6$, $BP = 4$, and $BC = 10$, find AC.

(A) $\sqrt{120}$ **(B)** $\sqrt{130}$ **(C)** $\sqrt{140}$ **(D)** $\sqrt{150}$ **(E)** $\sqrt{160}$

Solution: (A) Locate point Q on the opposite side of AB from C so that $APBQ$ is a parallelogram. It can be shown then that both $\angle QAC$ and $\angle QBC$ are right. By the Pythagorean Theorem, $QC^2 = QB^2 + BC^2 = QA^2 + AC^2 = AP^2 + BC^2 = BP^2 + AC^2 = 6^2 + 10^2 = 4^2 + AC^2$. Thus $AC^2 = 100 + 36 - 16 = 120$ and $AC = \sqrt{120}$.

17. If $a = \dfrac{1110}{1111}$, $b = \dfrac{2221}{2223}$, and $c = \dfrac{3331}{3334}$ which of the following is true?

(A) $a > b > c$ **(B)** $b > a > c$ **(C)** $c > a > b$
(D) $c > b > a$ **(E)** $b > c > a$

Solution: (E) Note that $\frac{1}{a} - 1 > \frac{1}{c} - 1 > \frac{1}{b} - 1$ so $b > c > a$.

18. In non-convex hexagon $ABCDEF$, $AB = BC = CD = DE = EF = FA$ and $\angle A \cong \angle C \cong \angle E$. If the degree measure common to these three angles is x, what is the degree measure of $\angle ABC$ in terms of x?

(A) $120 + x$ **(B)** $180 - x$ **(C)** $90 + 2x$ **(D)** $180 - 3x$ **(E)** $60 + 3x$

Solution: (A) Note that by SAS congruence, $\triangle BDE$ is equilateral. Of course, $\triangle ABF$ and $\triangle CBD$ are congruent isosceles triangles. Since all the angles around the point B add up to 360 degrees, we have

$$\angle ABC + 60 + 180 - x = 360$$

so that $\angle ABC = 360 - 180 - 60 + x = 120 + x$.

19. In how many ways can a 4×4 nailed down board be tiled by eight 1×2 dominoes? One way to tile the board is shown below.

(A) 16 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 49

Solution: (C) Split into cases. If no domino crosses the horizontal midline of the square there are 5^2 or 25 ways to tile, since there are 5 ways to tile a 2 by 4 rectangle. An even number of dominoes must cross the midline. If exactly two dominoes cross the midline there are 10 ways. These 10 further split as: just one way if the two that cross are not next to each other (they must be at opposite ends of the midline); four ways if they are next to each other far left (because of the five ways to do a 4 by 2 minus the one way that would lead to 4 dominoes crossing the midline); four ways if they are next to each other far right; and one way if the two occupy the center. Finally if four dominoes cross the midline, there is only one way. So there are $25 + 10 + 1 = 36$ ways to tile the 4 by 4 board with 1 by 2 dominoes.

20. Let L_1 and L_2 be two intersecting lines. Let P be an arbitrary point of the plane determined by L_1 and L_2. Consider the following sequence of transformations in this plane. First, the point P is reflected across line L_1 to point Q. Second, point Q is reflected across line L_2 to point R. This sequence of transformations that maps point P to point R is equivalent to

(A) a translation
(B) a reflection about some third line
(C) a rotation about the point of intersection of the lines by an angle equal to the smaller angle formed by the lines
(D) a rotation about the point of intersection of the lines by an angle equal to twice the smaller angle formed by the lines
(E) a translation followed by a reflection about some third line

Solution: (D) Since the composition of the sequence of transpositions preserves the distance to the point of intersection, it must either be a reflection or a rotation. Tracing out several points shows it cannot be a reflection across a point or line, so it must be a rotation about the point of intersection of the lines. Those examples also show the angle of rotation is twice the angle between the lines.

FALL 2011 MCNABB GDCTM CONTEST
ALGEBRA II SOLUTIONS

NO Calculators Allowed

1. A certain number is doubled. The result is then increased by nine. This result is decreased by 3. If this last number is 28, what was the original number?

   (A) -4 **(B)** 0 **(C)** 7 **(D)** 11 **(E)** 28

   Solution: (D) Work backwards: 28 to 31 to 22 to 11.

2. The expression
$$3\big((a+3b)4+2(5b+a)\big)$$
   is equivalent to the expression

   (A) $18a+22b$ **(B)** $15a+22b$ **(C)** $42a+42b$
   (D) $15a+66b$ **(E)** $18a+66b$

   Solution: (E) Distribute and collect: $3(4a+12b+10b+2a) = 3(6a+22b) = 18a+66b$.

3. How many of the numbers in this set below are irrational?
$$\sqrt{1.00}, \sqrt{1.01}, \sqrt{1.02}, \sqrt{1.03}, \cdots, \sqrt{3.98}, \sqrt{3.99}$$

   (A) 299 **(B)** 294 **(C)** 290 **(D)** 286 **(E)** 150

   Solution: (C) Of the 300 numbers in the list ten are rational, having the form $\sqrt{n^2/100}$ for $n = 10, 11, 12, \cdots, 19$.

4. If $f(3x+1) = \dfrac{2}{x+4}$, then $f(x+3) =$

   (A) $\dfrac{6}{x+14}$ **(B)** $\dfrac{6}{3x+14}$ **(C)** $\dfrac{6}{x+11}$ **(D)** $\dfrac{6}{x+17}$ **(E)** $\dfrac{2}{x+14}$

   Solution: (A) Replace x by $x-1/3$ in the first given equation to obtain $f(3x) = 2/(x+11/3) = 6/(3x+11)$. This gives $f(x) = 6/(x+11)$ so that $f(x+3) = 6/(x+14)$.

5. An ordered pair (m,n) of positive integers is called a *three-pair* if the interior angle of a regular polygon with m sides is three times the exterior angle of a regular polygon with n sides. How many *three-pairs* exist?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** more than 6

   Solution: (D) From $\dfrac{180(m-2)}{m} = 3\cdot\dfrac{360}{n}$ we solve for n, yielding $n = \dfrac{6m}{m-2}$. So $m = 3,4,5,6,8,14$ are the only positive integer values of m for which n is a positive integer greater than two. This follows because for $m > 14$, $< 6\dfrac{6m}{m-2} < 7$.

6. Amanda, Brice, and Carl all start working at SellMore with the same salary on the same day. They receive the following percent raises in their salary at the end of each of the first two years in order:

   Amanda: 5%, 3%

   Brice: 3%, 5%

   Carl: 4%, 4%

Which of them earns the most total over their first three years at SellMore?

(A) Amanda **(B)** Brice **(C)** Carl
(D) Amanda and Brice tied for the most **(E)** All three tied for the most

Solution: (A) Though the total sum of the percent raises is the same for all three employees, it is best to get the largest raise first as a simple calculation shows.

7. Let a, b, c, and d be non-zero constants. If the lines $ax + by = 0$ and $cx + dy = 0$ are perpendicular then which of these quantities must be zero?

(A) $ad - bc$ **(B)** $ac + bd$ **(C)** $ac - bd$
(D) $ad + bc$ **(E)** $a^2 + b^2 + c^2 + d^2$

Solution: (B) The slope of the first line equals $-a/b$. The opposite reciprocal of the slope of the second line equals d/c. From $-a/b = d/c$ we have $-ac = bd$ or $ac + bd = 0$.

8. If p people consume m pounds of mashed potato in h hours, then the pounds of mashed potato consumed by m people in p hours equals:

(A) mph **(B)** $\dfrac{m}{ph}$ **(C)** $\dfrac{m^2}{ph}$ **(D)** $\dfrac{m^2}{h}$ **(E)** $\dfrac{p^2}{m}$

Solution: (D) The pounds z consumed is jointly proportional to the number of people x and the number of hours y. From the given, $m = kph$ so that the constant of proportionality k equals $m/(ph)$. Thus $z = (m/(ph))mp = m^2/h$.

9. If $a < b < c < d < e$ which of the following must be true?

(A) $ab < cd$
(B) $c - a < e - c$
(C) $a^2 < e^2$
(D) $ad + bc < ac + bd$
(E) $b + d < 2c$

Solution: (D) (D) is equivalent to $ac - ad + bd - bc = (b - a)(d - c) > 0$.

10. Find the coefficient of x^{30} in the expansion of

$$(1 + x^3)(1 + x^6)(1 + x^9)(1 + x^{12}) \cdots (1 + x^{27})(1 + x^{30})$$

(A) 9 **(B)** 10 **(C)** 33 **(D)** 43 **(E)** 57

Solution: (B) After division by three all around, this problem is the same as asking in how many ways can 10 be written as the sum of distinct positive integers, including 10 itself, where order does not matter. The answer to this is also ten!.

11. In how many ways can the letters in the word *monsoon* be arranged so that the second n occurs before the third o?

(A) 210 **(B)** 216 **(C)** 252 **(D)** 256 **(E)** 260

Solution: (C) It is somewhat easier to count the complement. So we count the number of ways the second n occurs after the third o.

Case 1. The third o occurs at the third letter. $4 \cdot 3$ ways

Case 2. The third o occurs at the fourth letter. $3 \cdot 4 \cdot 3$ ways

Case 3. The third o occurs at the fifth letter. $\binom{4}{2} \cdot 4 \cdot 3 - \binom{4}{2} \cdot 1 \cdot 2!$ ways.

Case 4. The third o occurs at the sixth letter. $\binom{5}{2} \cdot 3!$ ways

As there are $7!/(3!2!) = 420$ ways to arrange *monsoon* altogether, our answer is $420 - 12 - 36 - 72 + 12 - 60 = 252$.

12. Suppose $a + b^{-1} = 4$ and $b + a^{-1} = 4/3$. If $a > b^{-1}$ what is the value of ab?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: (A) Multiply both equations together to obtain $ab + 2 + 1/ab = 16/3$ or $ab + 1/ab = 10/3$. By inspection or otherwise $ab = 3$.

13. In $\triangle ABC$, points E and F lie on $\overline{BC}$ and $\overline{AC}$ respectively. Let $\overline{AE}$ and $\overline{BF}$ intersect at G. If $\dfrac{AF}{FC} = \dfrac{3}{5}$ and G is the midpoint of $\overline{BF}$, then find the ratio $\dfrac{CE}{EB}$.

(A) 3 **(B)** 7/2 **(C)** 8/3 **(D)** 3/2 **(E)** 11/4

Solution: (C) Put masses of 5 at A, 3 at C and 8 at B. So $CE/EB = 8/3$.

14. Independently of each other, Hezy and Zeke each randomly pick a real number between 0 and 3. What is the probability that their choices differ by at least 1?

(A) 1/9 **(B)** 1/3 **(C)** 1/2 **(D)** 4/9 **(E)** 5/9

Solution: (D) Model by a square in the plane with vertices at $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$. The desired region is in this square and either above the line $y = x + 1$ or below the line $y = x - 1$. This region consists of two triangle which when glued together form a square of area 4. Thus the asked for probability equals 4/9.

15. Find the area enclosed by the graph of

$$|2y - 1| + |2y + 1| + 2|x| = 4$$

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4

Solution: (C) The equation graphs as a hexagon with vertices at

$$(0,1), (1,1/2), (1,-1/2), (0,-1), (-1,-1/2), (-1,1/2)$$

The y axis splits this hexagon into two congruent hexagons. So the area of the hexagon equals $2 \cdot (1/2) \cdot (2+1) \cdot 1 = 3$.

16. Find the sum

$$1 \cdot 25 + 2 \cdot 24 + 3 \cdot 23 + 4 \cdot 22 + \cdots + 24 \cdot 2 + 25 \cdot 1$$

(A) 2500 **(B)** 2725 **(C)** 2800 **(D)** 2825 **(E)** 2925

Solution: (E) The sum is $\binom{27}{3}$. Imagine the sum as the number of blocks in a structure with, working left to right, one column 25 blocks high, then two columns each 24 blocks high, etc.. Now read this by rows. The top row has 1 block. The second has 1+2 blocks. The third has 1+2+3 blocks,etc.. till the bottom row has $1 + 2 + 3 + 4 + \cdots + 25$ blocks. So this sum overall is the sum of the first 25 triangular numbers. Looking at the triangular numbers as a diagonal of Pascal's Triangle, and using the hook identity we see the sum is $\binom{27}{3} = 2925$. Can you find a purely combinatorial solution?

17. Square $ABCD$ of side-length 2 is inscribed in a circle. If chord AP bisects segment BC of the square, what is the square of the length of chord AP?

(A) 7 **(B)** 7.2 **(C)** 7.4 **(D)** 7.5 **(E)** 7.7

Solution: (B) Let chord AP bisect BC at Q. As $\angle ABC$ is right, then $AQ^2 = 2^2 + 1^2$ so $AQ = \sqrt{5}$. Then use power of a point at Q to show that $\sqrt{5} \cdot QP = 1 \cdot 1$. Thus $QP = 1/\sqrt{5}$. Hence $AP^2 = (\sqrt{5} + 1/\sqrt{5})^2 = 5 + 2 + 1/5 = 7.2$.

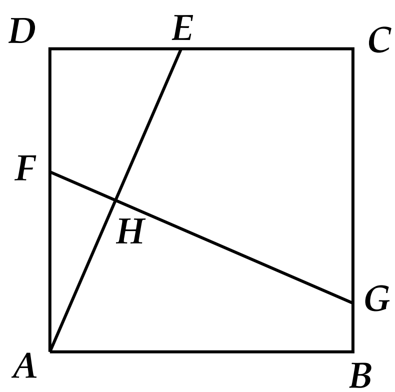
18. Let $f(x) = ax^2 + bx + c$, where a, b, and c are all non-zero constants. If $c = \dfrac{b^2}{4a}$, then the graph of f must

(A) be symmetric with respect to the y axis
(B) be symmetric with respect to the x axis
(C) be tangent to the x axis
(D) be tangent to the y axis
(E) have a maximum point

Solution: (C) The given condition means that the discriminant is zero, so the function f has a double real root, which must be at the vertex of the parabola.

19. Inside square $ABCD$ point E lies on side $\overline{CD}$ with $\dfrac{CE}{ED} = \dfrac{5}{3}$. The perpendicular bisector of $\overline{AE}$ intersects the square at points F and G and intersects $\overline{AE}$ at H, as shown. Find the ratio $\dfrac{FH}{HG}$

(A) 3/13 **(B)** 1/4 **(C)** 2/7 **(D)** 1/3 **(E)** 4/11



Solution: (A) Draw the line through point H parallel to CD, intersecting DA at J and BC at K. Then $\triangle FHJ \sim \triangle GHK$. Assume $DE = 6$ and $EC = 10$. By the Midline Theorem $HJ = 3$ and $HK = 16 - 3 = 13$. From the similarity, the result follows.

20. In acute triangle ABC, the intersection of its three altitudes, called the *orthocenter*, is labeled P. Given that $AP = 6$, $BP = 4$, and $BC = 10$, find AC.

(A) $\sqrt{120}$ **(B)** $\sqrt{130}$ **(C)** $\sqrt{140}$ **(D)** $\sqrt{150}$ **(E)** $\sqrt{160}$

Solution: (A) Locate point Q on the opposite side of AB from C so that $APBQ$ is a parallelogram. It can be shown then that both $\angle QAC$ and $\angle QBC$ are right. By the Pythagorean Theorem, $QC^2 = QB^2 + BC^2 = QA^2 + AC^2 = AP^2 + BC^2 = BP^2 + AC^2 = 6^2 + 10^2 = 4^2 + AC^2$. Thus $AC^2 = 100 + 36 - 16 = 120$ and $AC = \sqrt{120}$.

Fall 2011 McNabb GDCTM Contest Solutions Pre-Calculus

NO Calculators Allowed

1. The expression
$$3\big((a+3b)4+2(5b+a)\big)$$
is equivalent to the expression

   (A) $18a+22b$ **(B)** $15a+22b$ **(C)** $42a+42b$
   (D) $15a+66b$ **(E)** $18a+66b$

   Solution: (E) Distribute and collect: $3(4a+12b+10b+2a) = 3(6a+22b) = 18a+66b$.

2. If a raindrop has a volume of 10 cubic millimeters, a certain school yard has dimensions 50 meters by 40 meters, and this yard receives 5 centimeters of rain, the number of raindrops that fell on the yard is

   (A) 10^9 **(B)** 10^{10} **(C)** 10^{11} **(D)** 10^{12} **(E)** 10^{13}

   Solution: (B) Find total volume of rain in cubic millimeters and divide by 10, so that
$$\frac{50\cdot 10^3\cdot 40\cdot 10^3\cdot 50}{10} = 5\cdot 5\cdot 4\cdot 10^1\cdot 10^3\cdot 10^1\cdot 10^3 = 100\cdot 10^8 = 10^{10}$$

3. Recall that $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x^2]-[x]^2$, find $f(\pi)$.

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

   Solution: (C) Since $9<\pi^2<10$, then $[\pi^2]=9$. And $[\pi]^2 = 3^2 = 9$.

4. How many subsets of $\{a,b,c,d,e\}$ have an odd number of elements?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

   Solution: (E) Add every other element of the fifth row of Pascal's Triangle: $5+10+1=16$.

5. How many perfect squares are in the sequence of integers
$$1, 11, 111, 1111, 11111, 111111, \cdots$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

   Solution: (B) When a perfect square is divided by 4 the remainder can be only 0 or 1. Thus every integer ending in 11 cannot be a perfect square. So only the number 1 in this list is a perfect square.

6. Suppose $a+b^{-1}=4$ and $b+a^{-1}=4/3$. If $a>b^{-1}$ what is the value of ab?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: (A) Multiply both equations together to obtain $ab+2+1/ab = 16/3$ or $ab+1/ab = 10/3$. By inspection or otherwise $ab=3$.

7. If $f(3x+1) = \dfrac{2}{x+4}$, then $f(x+3) =$

(A) $\dfrac{6}{x+14}$ **(B)** $\dfrac{6}{3x+14}$ **(C)** $\dfrac{6}{x+11}$ **(D)** $\dfrac{6}{x+17}$ **(E)** $\dfrac{2}{x+14}$

Solution: (A) Replace x by $x - 1/3$ in the first given equation to obtain $f(3x) = 2/(x+11/3) = 6/(3x+11)$. This gives $f(x) = 6/(x+11)$ so that $f(x+3) = 6/(x+14)$.

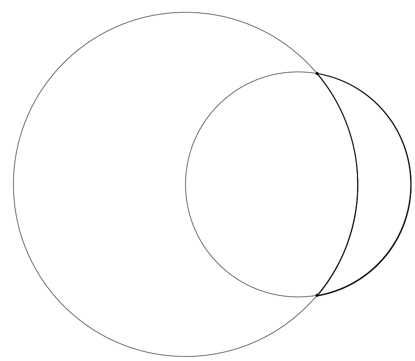
8. Independently of each other, Hezy and Zeke each randomly pick a real number between 0 and 3. What is the probability that their choices differ by at least 1?

(A) 1/9 **(B)** 1/3 **(C)** 1/2 **(D)** 4/9 **(E)** 5/9

Solution: (D) Model by a square in the plane with vertices at $(0,0)$, $(3,0)$, $(3,3)$, and $(0,3)$. The desired region is in this square and either above the line $y = x+1$ or below the line $y = x-1$. This region consists of two triangle which when glued together form a square of area 4. Thus the asked for probability equals 4/9.

9. The distance between the two centers of the two circles is 3, the center of the larger circle lies on the smaller circle, and the points of intersection of the two circles lie on the same diameter of the smaller circle. Find the area of that part of the smaller circle that lies outside of the larger circle.

(A) 9 **(B)** $\pi/4$ **(C)** $9\pi/8 + 3\sqrt{2}/2$ **(D)** $3\pi - 3$ **(E)** 10



Solution: (A) The radius of the smaller circle is 3 and the radius of the larger circle is $3\sqrt{2}$. The desired area is the difference of the area of a semicircle of the smaller circle and a region that is itself the difference between a quarter circle of the larger circle and a right isosceles triangle with leg equal to the radius of the larger circle. Thus we obtain our answer is

$$9\pi/2 - (18\pi/4 - (1/2)18) = 18/2 = 9$$

10. What is the positive difference between the largest and smallest real solutions of

$$x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: (E) Use the Rational Root Theorem to guess rational roots. You can quickly see that $x = 1$ is root. Then that it is a double root. This brings you down to $x^2 + 6x + 9 = (x+3)^2$, so that $x = -3$ is the other real root. So $1 - (-3) = 4$ is the answer.

11. Let $f(x) = ax^2 + bx + c$, where a, b, and c are all non-zero constants. If $c = \dfrac{b^2}{4a}$, then the graph of f must

(A) be symmetric with respect to the y axis
(B) be symmetric with respect to the x axis
(C) be tangent to the x axis
(D) be tangent to the y axis
(E) have a maximum point

Solution: (C) The given condition means that the discriminant is zero, so the function f has a double real root, which must be at the vertex of the parabola.

12. In acute triangle ABC, the intersection of its three altitudes, called the *orthocenter*, is labeled P. Given that $AP = 6$, $BP = 4$, and $BC = 10$, find AC.

(A) $\sqrt{120}$ **(B)** $\sqrt{130}$ **(C)** $\sqrt{140}$ **(D)** $\sqrt{150}$ **(E)** $\sqrt{160}$

Solution: (A) Locate point Q on the opposite side of AB from C so that $APBQ$ is a parallelogram. It can be shown then that both $\angle QAC$ and $\angle QBC$ are right. By the Pythagorean Theorem, $QC^2 = QB^2 + BC^2 = QA^2 + AC^2 = AP^2 + BC^2 = BP^2 + AC^2 = 6^2 + 10^2 = 4^2 + AC^2$. Thus $AC^2 = 100 + 36 - 16 = 120$ and $AC = \sqrt{120}$.

13. A certain polynomial $P(x)$ has the property that all its coefficients are non-negative integers, none of which is larger than 6. If $P(7) = 2011$, what is P's coefficient of x^2?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution: (E) It follows that $P(x) = ax^3 + bx^2 + cx + d$ where $abcd$ is the base 7 representation of 2011. So that $abcd = 5602$, so that $b = 6$.

14. Square $ABCD$ of side-length 2 is inscribed in a circle. If chord AP bisects segment BC of the square, what is the square of the length of chord AP?

(A) 7 **(B)** 7.2 **(C)** 7.4 **(D)** 7.5 **(E)** 7.7

Solution: (B) Let chord AP bisect BC at Q. As $\angle ABC$ is right, then $AQ^2 = 2^2 + 1^2$ so $AQ = \sqrt{5}$. Then use power of a point at Q to show that $\sqrt{5} \cdot QP = 1 \cdot 1$. Thus $QP = 1/\sqrt{5}$. Hence $AP^2 = (\sqrt{5} + 1/\sqrt{5})^2 = 5 + 2 + 1/5 = 7.2$.

15. Five horses are in a race. In how many ways can they finish if ties are allowed?

(A) 511 **(B)** 530 **(C)** 531 **(D)** 541 **(E)** 625

Solution: (D) This can be done by cases.

(i) No ties. $5! = 120$ ways

(ii) Two horses tie. $\binom{5}{2} \cdot 4 \cdot 3! = 240$ ways

(iii) Three horses tie. $\binom{5}{3} \cdot 3 \cdot 2! = 60$ ways

(iv) Four horses tie. $\binom{5}{4} \cdot 2 = 10$ ways

(v) Five horses tie. 1 way

(vi) Two horses tie and another two horses tie. $3 \cdot \binom{5}{2} \cdot \binom{3}{2} = 90$ ways

(vii) Three horses tie and the remaining two horses tie. $2 \cdot \binom{5}{3} = 20$ ways

So altogether there are $120 + 240 + 60 + 10 + 1 + 90 + 20 = 541$ ways for 5 horses to tie.

16. Find the coefficient of x^{30} in the expansion of

$$(1+x^3)(1+x^6)(1+x^9)(1+x^{12})\cdots(1+x^{27})(1+x^{30})$$

(A) 9 **(B)** 10 **(C)** 33 **(D)** 43 **(E)** 57

Solution: (B) After division by three all around, this problem is the same as asking in how many ways can 10 be written as the sum of distinct positive integers, including 10 itself, where order does not matter. The answer to this is also ten!.

17. The graph of $y = ax^3 + bx^2 + cx + d$ is shown below. Which of the following must be true?

I. $a < 0$

II. $c < 0$

III. $cd > 0$

(A) I only **(B)** II only **(C)** I and II only
(D) II and III only **(E)** I,II, and III

Solution: (C) By the end behaviour of the cubic, $a < 0$. Near $x = 0$ the cubic behaves like the line $cx + d$, so that $c < 0$ and $d > 0$. So I and II are true, but III is false.

18. Find the radius of the sphere that contains the points $(0, -3, -2)$, $(0, -3, 2)$, $(2, 3, 1)$, and $(-2, 3, 1)$.

(A) $\frac{43}{12}$ **(B)** $\frac{\sqrt{1945}}{12}$ **(C)** $\frac{47}{12}$ **(D)** $\frac{\sqrt{2011}}{12}$ **(E)** $\frac{49}{12}$

Solution: (B) If two points lie on a sphere, the plane that bisects the segment with those two points as endpoints contains the center of the sphere. The bisecting plane for the first two points is $z = 0$. The bisecting plane for the last two points is $x = 0$. Thus the center of the sphere lies on the y axis; let's say it is $(0, y, 0)$. The square of the distance from this center to the first point is $(y+3)^2 + 4$. The square of the distance from this center to the third point is $4 + (y-3)^2 + 1$. So that $(y+3)^2 + 4 = 4 + (y-3)^2 + 1$. Solving this gives $y = 1/12$. Plugging back in, the square of the radius is $1945/144$.

19. How many solutions are there to the equation $(1.1)^x = \log_{1.1} x$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: (C) Note $(1.1)^2 = 1.21$ while $\log_{1.1} 2 > 2$. Thus the graph of the function $\log_{1.1} x$ is higher than the graph of $(1.1)^x$ at $x = 2$. Yet for $0 < x < 1$ the graph of $\log_{1.1} x$ lies below the x axis. So one solution of the equation lies in the interval $(0, 2)$. Since $\log_{1.1} x$ gets less steep as x increases while $(1.1)^x$ gets more steep, there is one more solution of the equation satisfying $x > 2$. Thus there are 2 solutions altogether.

20. If $x^4 + \frac{1}{x^4} = 194$, then $x^6 + \frac{1}{x^6}$ could be equal to

(A) 2702 **(B)** 2730 **(C)** 3088 **(D)** 3090 **(E)** 3102

Solution: (A) If $x^2 + 1/x^2 = a$ then $x^4 + 1/x^4 = a^2 - 2 = 194$, so that $a = 14$. Then $x^6 + 1/x^6 = (x^4 + 1/x^4)(x^2 + 1/x^2) - (x^2 + 1/x^2) = 194 \cdot 14 - 14 = 193 \cdot 14 = 2702$.

Fall 2011 McNabb GDCTM Contest Solutions Calculus

NO Calculators Allowed

All variables are assumed to represent real numbers unless stated in the problem otherwise.

1. The number of even factors of 7^7 is

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

   Solution: (A) Since 7^7 is odd it has no even factors.

2. Recall that $[x]$ denotes the greatest integer less than or equal to x. If $f(x) = [x^2] - [x]^2$, find $f(\pi)$.

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

   Solution: (C) Since $9 < \pi^2 < 10$, then $[\pi^2] = 9$. And $[\pi]^2 = 3^2 = 9$.

3. How many subsets of $\{a, b, c, d, e\}$ have an odd number of elements?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

   Solution: (E) Add every other element of the fifth row of Pascal's Triangle: $5 + 10 + 1 = 16$.

4. How many solutions are there to the equation $2^a = a^2$?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: (D) There are two positive solutions, when $a = 2$ and when $a = 4$. There is exactly one negative solution. That these are all can be made clear from the graph and reasoning about concave up functions.

5. Given that the piecewise function

$$f(x) = \begin{cases} 4x & \text{if } x \leq 0 \\ ax^2 + bx + c & \text{if } 0 < x < 1 \\ 6 - 3x & \text{if } x \geq 1 \end{cases}$$

   is differentiable at all real numbers, find the value of $a + b + c$.

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Solution: (C) Since the function is differentiable it is also continuous. Thus $a + b + c$ equals the value of $ax^2 + bx + c$ when $x = 1$, so that it also equals the value of $6 - 3x$ when $x = 1$, that is, 3.

6. The value of $13\sin(\tan^{-1}(5/12)) + 15\sin(\tan^{-1}(9/12))$ is

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Solution: (D) This can be solved by looking at a 13, 14,15 triangle. Or just use the definitions directly with a 5,12,13 right triangle and a 9,12,15 right triangle. Then we get $13(5/13) + 15(9/15) = 14$.

7. Let $f(x)$ be differentiable at $x = 3$. If $f(3) = 5$ and $\left(\frac{1}{f}\right)'(3) = 4$ then what is the value of $f'(3)$?

(A) -100 **(B)** -20 **(C)** -1/4 **(D)** 1/4 **(E)** 20

Solution: (A) By the reciprocal rule, $(1/f)'(3) = -f'(3)/f^2(3) = -f'(3)/25 = 4$, so that $f'(3) = 4(-25) = -100$.

8. Determine the value of

$$\lim_{x\to 0}(\cos x)^{\frac{1}{\sin^2 x}}$$

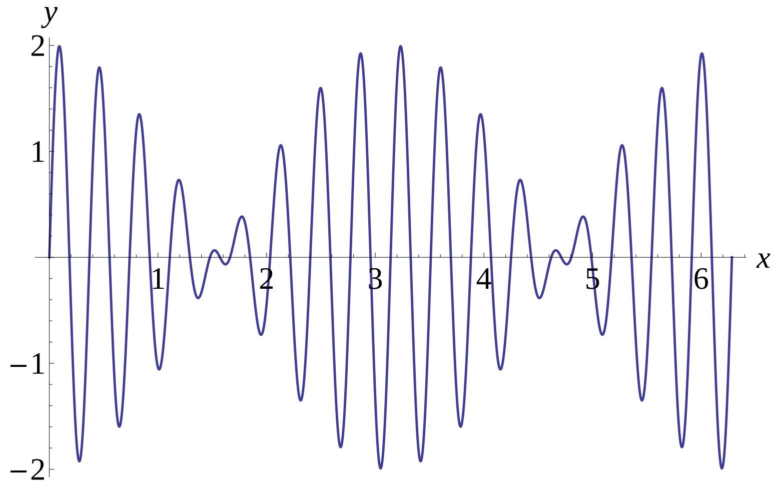
(A) 0 **(B)** 1 **(C)** e **(D)** e^2 **(E)** $e^{-\frac{1}{2}}$

Solution: (E) Apply L'Hospital's rule. Note $\lim_{x\to 0}\dfrac{\ln(\cos x)}{\sin^2 x} = \lim_{x\to 0}\dfrac{-1}{2\cos^2 x} = -1/2$.

9. The graph of $y = \sin ax + \sin bx$ is shown below for x in the interval $[0, 2\pi]$. Given that a and b are positive integers, with $a + b$ large compared to $a - b$ and $a > b$, the value of $a - b$ could be:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

Solution: (B) By an identity, $\sin ax + \sin bx = 2\sin\left(\left(\frac{a+b}{2}\right)x\right)\cos\left(\left(\frac{a-b}{2}\right)x\right)$, we can view the cosine factor as modulating the amplitude of the sine wave. From the diagram we see the cosine must be just the regular $\cos x$. Thus $a - b = 2$.



10. An eight by eight matrix has its (i, j)th entry given by $f(i)g(j)$ where $f(i) = 4i - 3$ and $g(j) = 2j + 5$. What is the sum of all of the entries of this matrix?

(A) 13440 **(B)** 13540 **(C)** 13640 **(D)** 13740 **(E)** 13840

Solution: (A) Using the distributive property the sum asked for is the same as

$$\left(\sum_{i=1}^{8} 4i - 3\right)\left(\sum_{j=1}^{8} 2j + 5\right)$$

or $((1 + 29)/2) \cdot 8 \cdot ((7 + 21)/2) \cdot 8 = 30 \cdot 28 \cdot 16 = 840 \cdot 16 = 13440$.

11. If $f(x) = x^7 + x$, then what is the value of the second derivative of the inverse function of f at 2? That is, what is $(f^{-1})''(2)$?

(A) -7/128 **(B)** -7/16 **(C)** -21/256 **(D)** -3/64 **(E)** 3/64

Solution: (C) By the Inverse Function Theorem, the first derivative of the inverse is $1/f'(f^{-1}(x))$. Differentiating again, using the Reciprocal Rule, the Chain Rule, and the Inverse Function Theorem again we obtain

$$\left(f^{-1}\right)''(x) = -\frac{f''(f^{-1}x)}{(f'(f^{-1}(x)))^3}$$

Since $f^{-1}(2) = 1$,$f'(x) = 7x^6 + 1$, and $f''(x) = 42x^5$, it follows that $\left(f^{-1}\right)''(2) = -42/(7 + 1)^3 = -42/512 = -21/256$.

12. Find the absolute maximum value of $f(x) = \sin^8 x \cos^4 x$ on the interval $[0, 2\pi]$.

(A) $\dfrac{2^4}{3^6}$ **(B)** $\dfrac{1}{2^6}$ **(C)** $\dfrac{3^4}{2^{12}}$ **(D)** $\dfrac{3^6}{2^{10}}$ **(E)** $\dfrac{3^2}{2^6}$

Solution: (A) Change variables with $u = \sin x$ and $1 - u^2 = \cos^2 x$. Then $f(x) = g(u) = u^8(1-u^2)^2$. Then by the Product Rule, $g'(u) = 4u^7(1-u^2)(2-3u^2)$. The critical points are $u = 0, u^2 = 2/3$ and the endpoints are $u = -1, 1$. Plugging all these in to g we see the maximum value of g, which equals the maximum value of f, occurs when $u^2 = 2/3$. The maximum value itself equals $(2/3)^4(1/3)^2 = 2^4/3^6$.

13. Five horses are in a race. In how many ways can they finish if ties are allowed?

(A) 511 **(B)** 530 **(C)** 531 **(D)** 541 **(E)** 625

Solution: (D) This can be done by cases.

(i) No ties. $5! = 120$ ways

(ii) Two horses tie. $\binom{5}{2} \cdot 4 \cdot 3! = 240$ ways

(iii) Three horses tie. $\binom{5}{3} \cdot 3 \cdot 2! = 60$ ways

(iv) Four horses tie. $\binom{5}{4} \cdot 2 = 10$ ways

(v) Five horses tie. 1 way

(vi) Two horses tie and another two horses tie. $3 \cdot \binom{5}{2} \cdot \binom{3}{2} = 90$ ways

(vii) Three horses tie and the remaining two horses tie. $2 \cdot \binom{5}{3} = 20$ ways

So altogether there are $120 + 240 + 60 + 10 + 1 + 90 + 20 = 541$ ways for 5 horses to tie.

14. For the sequence given by $t_{n+1} = \dfrac{t_n + t_{n-1} + 1}{t_{n-2}}$, with $t_1 = 4$, $t_2 = 2$ and $t_3 = 5$, find t_{2011}.

(A) 4 **(B)** 2 **(C)** 5 **(D)** 7/5 **(E)** 16/5

Solution: (C) Applying the recursion relation we extend the sequence as far as (from the beginning)

$$4, 2, 5, 2, 4, 7/5, 16/5, 7/5, 4, 2, 5, etc....$$

So this sequence repeats with a cycle length of 8. Dividing 2011 by 8 gives a remainder of 3. The third element of the sequence is 5.

15. Find the radius of the sphere that contains the points $(0, -3, -2)$, $(0, -3, 2)$, $(2, 3, 1)$, and $(-2, 3, 1)$.

(A) $\dfrac{43}{12}$ **(B)** $\dfrac{\sqrt{1945}}{12}$ **(C)** $\dfrac{47}{12}$ **(D)** $\dfrac{\sqrt{2011}}{12}$ **(E)** $\dfrac{49}{12}$

Solution: (B) If two points lie on a sphere, the plane that bisects the segment with those two points as endpoints contains the center of the sphere. The bisecting plane for the first two points is $z = 0$. The bisecting plane for the last two points is $x = 0$. Thus the center of the sphere lies on the y axis; let's say it is $(0, y, 0)$. The square of the distance from this center to the first point is $(y+3)^2 + 4$. The square of the distance from this center to the third point is $4 + (y-3)^2 + 1$. So that $(y+3)^2 + 4 = 4 + (y-3)^2 + 1$. Solving this gives $y = 1/12$. Plugging back in, the square of the radius is $1945/144$.

16. A certain polynomial $P(x)$ has the property that all its coefficients are non-negative integers, none of which is larger than 6. If $P(7) = 2011$, what is P's coefficient of x^2?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution: (E) It follows that $P(x) = ax^3 + bx^2 + cx + d$ where $abcd$ is the base 7 representation of 2011. So that $abcd = 5602$, so that $b = 6$.

17. If $a = \dfrac{1110}{1111}$, $b = \dfrac{2221}{2223}$, and $c = \dfrac{3331}{3334}$ which of the following is true?

(A) $a > b > c$ **(B)** $b > a > c$ **(C)** $c > a > b$ **(D)** $c > b > a$ **(E)** $b > c > a$

Solution: (E) Note that $\frac{1}{a} - 1 > \frac{1}{c} - 1 > \frac{1}{b} - 1$ so $b > c > a$.

18. Which of the following are always true about the pair of functions $y = a^x$ and $y = \log_a x$, where $a > 1$?

I. Both are increasing on their domains

II. For the same $a > 1$, their graphs never intersect

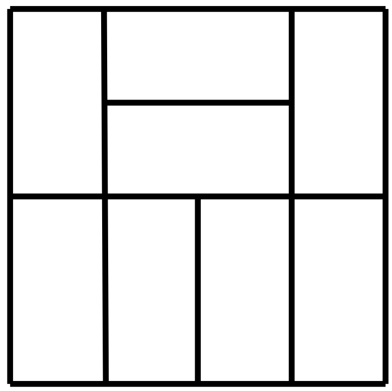
III. For the same $a > 1$, they are inverse functions of each other

(A) I only **(B)** II only **(C)** I and II only
(D) I and III only **(E)** I, II and III

Solution: (D) Choices I and III are always true. But II is false for values of a close enough to 1, such as $a = 1.1$.

19. In how many ways can a 4×4 nailed down board be tiled by eight 1×2 dominoes? One way to tile the board is shown below.

(A) 16 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 49



Solution: (C) Split into cases. If no domino crosses the horizontal midline of the square there are 5^2 or 25 ways to tile, since there are 5 ways to tile a 2 by 4 rectangle. An even number of dominoes must cross the midline. If exactly two dominoes cross the midline there are 10 ways. These 10 further split as: just one way if the two that cross are not next to each other (they must be at opposite ends of the midline); four ways if they are next to each other far left (because of the five ways to do a 4 by 2 minus the one way that would lead to 4 dominoes crossing the midline); four ways if they are next to each other far right; and one way if the two occupy the center. Finally if four dominoes cross the midline, there is only one way. So there are $25 + 10 + 1 = 36$ ways to tile the 4 by 4 board with 1 by 2 dominoes.

20. If $a + b + c = 9$ and $ab + bc + ca = 7$ then the maximum possible value of c is closest to

(A) 8 **(B)** 8.5 **(C)** 9 **(D)** 9.5 **(E)** 10

Solution: (A) Let a, b, and c be the real roots of a cubic polynomial $f(x) = x^3 - 9x^2 + 7x + k$ with $a \le b \le c$. All k does is shift the graph of f up or down. The largest value of c will occur when k is as small as possible so that f still has all real roots. This occurs when f has a double real root, that is $a = b$. And that happens when f and $f' = 3x^2 - 18x + 7$ have a common root. The roots of f' are $3 \pm 2\sqrt{15}/3$. The one to be shared with f is the lesser because this choice maximizes c. We now know that $a = b = 3 - 2\sqrt{15}/3$. From $a + b + c = 9$ we have $c = 9 - (6 - 4\sqrt{15}/3) = 3 + 4\sqrt{15}/3 \approx 3 + 4(4 - 1/8)/3 \approx 8.17$. So 8 is the closest to this maximum value.