Fall 2014 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. Express 188 as the sum of two prime numbers.
   Answer: $181 + 7$ **or four others.**
   Others are: $109 + 79$, $157 + 31$, $151 + 37$,and $127 + 61$.

2. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?
   Answer: 5.22
   The 8 Fiji apples cost $4.68 + 2(.90) = 6.48$ while the 9 Gala apples cost $5.39 + 3(.97) = 8.30$. So the change is $20 - 14.78 = 5.22$.

3. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks 4/3 as fast as Jill. When they meet, how far is Jack from where Jill started?
   Answer: 300 **meters.**
   If Jill walks x meters then Jack walks $4x/3$ in the same time. From $4x/3 + x = 700$ we get $x = 300$, so Jack is 300 meters from where she started when they meet.

4. A number of students pitch in to buy a gift for their teacher. If each pays 8 dollars, the total collected would be too great by 3 dollars. If each pays 7 dollars, the total collected would be too little by 4 dollars. How much does the gift cost?
   Answer: 53**.**
   The difference of the over and under is 7 dollars so there must be 7 students. Then $56 - 3 = 53$ is the amount of the gift.

5. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.
   Answer: acd/b
   In the answer each letter appears once, either as a factor in the numerator or the denominator. Suppose a were to increase while all other letters remain fixed. Then the tea leaves have become weaker so more are needed to fill the thermos's. So a is in the numerator. Suppose b increases. Then the tea leaves have become stronger, so b is in the denominator. If c increases the thermos's are now larger, so c is in the numerator. If d increases more tea must be made, so d is in the numeratory. Thus the answer is acd/b.

6. Forty-one erasers are distributed to n students. If at least one student always receives at least 6 erasers no matter how the erasers are distributed, what is the largest possible value of n?
   Answer: 8
   If there are 8 students and we distribute 5 erasers to each, one more eraser must be distributed, so whoever gets that eraser now has 6 erasers. So $n \geq 8$. But if there are nine

students it is easy to distribute 5 erasers each to 8 of them and the last eraser to the ninth student. Thus the greatest possible value of n is 8.

7. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?
Answer: 1122**.**
Certainly 1122 works since 13 goes into it 86 times with 4 left over, while 19 goes into it 59 times with 1 left over. Any smaller number that works is obtained by taking a 13 by 19 block of soldiers out (since that block can be rotated to fit either way, and no smaller block like that exists). Since $13 \cdot 19 > 122$ we have got the smallest brigade size over 1000.

8. Suppose that m and n are positive integers satisfying
$$\frac{1}{77} = \frac{n}{7} - \frac{3}{m}$$
Find the value of $m + n$.
Answer: 13
Since $1/77 = 2/7 - 3/11$ works we must show this in the only solution in positive integers. The condition given is equivalent to $m = (11 \cdot 7 \cdot 3)/(11n - 1)$. Since the 11 is of no help, $11n - 1$ must be a factor of 21. So the only positive solution is $n = 2$ which forces $m = 11$.

9. Find the smallest possible value of $ab + cd + ef$ if each letter stands for a distinct element of the set $\{1, 2, 3, 4, 5, 6\}$.
Answer: 28
Note that $28 = 1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4$. Other choices increase the sum. The closest is $1 \cdot 6 + 3 \cdot 5 + 2 \cdot 4 = 29$.

10. There are 5 yellow balls, 8 red balls, and 7 green balls in a bag. What is the minimum number of balls that must be drawn to guarantee that at least 6 of them are the same color?
Answer: 16**.**
If fifteen balls are drawn it could be that 5 of them are yellow, 5 red, and 5 green. With sixteen balls, even if you make 5 of them of each color, there is still one ball left to be assigned, so some color must end up with 6 balls.

11. The pages of the book *Science of Mechanics in the Middle Ages* are numbered from 1 to 711. Considering all the digits needed to print these page numbers starting from page 1, on what page number does the 241st '1'occur?
Answer: 701
Pages 1 thru 9 use 1 one. The pages 10 thru 99 use 19 ones. The pages 100 thru 700 use $100 + 60 + 60 = 220$ ones.

12. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1 + \sqrt{3}, \qquad 2 - \sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

Answer: $2-\sqrt{2}, \sqrt{2}/2, 1+\sqrt{3}, 2\sqrt{2}$
The statement $2-\sqrt{2} < \sqrt{2}/2$ is equivalent to $\sqrt{2} > 4/3$ which is true. Also, the statement $1+\sqrt{3} < 2\sqrt{2}$ is equivalent to the true statement $2\sqrt{24}+1 < 11$.

13. Find the sum of the positive even factors of 10000.
Answer: 23430
Since $10^4 = 2^4 \cdot 5^4$ the required number is equal to

$$(2+2^2+2^3+2^4)(1+5+5^2+5^3+5^4)$$

14. In how many ways can 4 different rings be placed on the four fingers of the right hand? Here the order of the rings on a given finger matters and each finger can accomodate all four rings.
Answer: 840
Notation: $a-b-c-d$ means one finger gets a rings, another b rings, etc..
Case I. $4-0-0-0$ can occur in $4 \cdot 4! = 96$ ways (as in 4 choices as to which finger gets all four rings, and then the number of ways to permute the 4 rings on that finger.).
Case II. $3-1-0-0$ can occur in $4 \cdot 4 \cdot 3! \cdot 3 = 288$ ways. The first 4 if for choosing which 3 rings go together. The second 4 is for choosing which finger to put them on. The 3! is for permuting those three rings on that finger. The 3 is for choosing which of the remaining three fingers to place the final ring on.
Case III. $2-2-0-0$ can occur in $3\dot{4} \cdot 2 \cdot 3 \cdot 2 = 144$ ways. The first 3 is for choosing who goes with ring A. (Label the rings say A, B, C, D.) The four is for choosing which finger A and its match go on. The two is for permuting A and its match. The 3 is for choosing which remaining finger to put the last two rings on. The two is for permuting those last two rings.
Case IV. $2-1-1-0$ can occur in $6 \cdot 4 \cdot 2 \cdot 3 \cdot 2 = 288$ ways. The six is for choosing which two rings will be paired. The four is for choosing which finger to put the pair on. The two is for permuting that pair. There are now two rings left, call them C and D. The three is for choosing which of three fingers to put C on. The two is for choosing which of two fingers to put D on.
Case V. $1-1-1-1$ can occur in 4! ways.
And $96+288+144+288+24 = 840$.
OR
The distribution of the number of rings assigned to each finger can be counted by stars and bars as $\binom{7}{3} = 35$. Then multiply this by $4! = 24$ as the four rings are distinct. Hence, $35 \cdot 24 = 840$.

15. Reading right to left, what is the first non-zero digit of 25!?
Answer: 4
After casting out the six matching factors of 2 and 5, the unit digits of the factors left are

$$3,3,6,7,8,9,1,2,3,4,6,7,8,9,1,2,3,4$$

It is not so hard to see what the unit's digit of the product of these factors will be, using tricks like $7 \cdot 3$ ends in a one, $9 \cdot 9$ ends in a one, 4! ends in a four, etc....

Fall 2014 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?
   Answer: 5.22
   The 8 Fiji apples cost $4.68 + 2(.90) = 6.48$ while the 9 Gala apples cost $5.39 + 3(.97) = 8.30$. So the change is $20 - 14.78 = 5.22$.

2. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.
   Answer: acd/b
   In the answer each letter appears once, either as a factor in the numerator or the denominator. Suppose a were to increase while all other letters remain fixed. Then the tea leaves have become weaker so more are needed to fill the thermos's. So a is in the numerator. Suppose b increases. Then the tea leaves have become stronger, so b is in the denominator. If c increases the thermos's are now larger, so c is in the numerator. If d increases more tea must be made, so d is in the numeratory. Thus the answer is acd/b.

3. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?
   Answer: 1122**.**
   Certainly 1122 works since 13 goes into it 86 times with 4 left over, while 19 goes into it 59 times with 1 left over. Any smaller number that works is obtained by taking a 13 by 19 block of soldiers out (since that block can be rotated to fit either way, and no smaller block like that exists). Since $13 \cdot 19 > 122$ we have got the smallest brigade size over 1000.

4. Suppose that m and n are positive integers satisfying
$$\frac{1}{77} = \frac{n}{7} - \frac{3}{m}$$
   Find the value of $m + n$.
   Answer: 13
   Since $1/77 = 2/7 - 3/11$ works we must show this in the only solution in positive integers. The condition given is equivalent to $m = (11 \cdot 7 \cdot 3)/(11n - 1)$. Since the 11 is of no help, $11n - 1$ must be a factor of 21. So the only positive solution is $n = 2$ which forces $m = 11$.

5. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1+\sqrt{3}, \qquad 2-\sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$
   Answer: $2-\sqrt{2}, \sqrt{2}/2, 1+\sqrt{3}, 2\sqrt{2}$

The statement $2-\sqrt{2}<\sqrt{2}/2$ is equivalent to $\sqrt{2}>4/3$ which is true. Also, the statement $1+\sqrt{3}<2\sqrt{2}$ is equivalent to the true statement $2\sqrt{24}+1<11$.

6. Recall that $\binom{m}{n}$ stands for the number of ways of choosing n objects from a set of m objects. Name a solution m greater than 2 of the equation

$$\binom{m+2}{3}=4\binom{m}{2}$$

Answer: $m=7$
The stated condition is

$$\frac{(m+2)(m+1)m}{6}=\frac{4m(m-1)}{2}$$

which simplifies to $m^2-9m+14=(m-2)(m-7)=0$.

7. Simplify

$$x-y-(3x-4y)-(x-y-(3x-4y))$$

Answer: 0

8. How many arrangements of the letters in GDCTM do not have any 3 consecutive letters in alphabetical order? So, for instance, you would count DGCTM but you would not count DGCMT.
Answer: 70
Inclusion-Exclusion works well here, with an important catch. Count the opposite case, where 3 or more consecutive letters are in alphabetical order. This works as:

$$\binom{5}{3}\cdot 3\cdot 2-\binom{5}{4}\cdot 2=50$$

so that the number we seek is $120-50=70$.
Note in the first term of Inclusion-Exclusion we are counting places where three consecutive letters are in alphabetical order occur. First we pick which 3 letters will be in alphabetical order. Then we choose which of 3 places we can lay them down. Then we pick the order of the remaining two letters.
In the second term of Inclusion-Exclusion we are counting places where four consecutive letters are in alphabetical order occur. First we pick which 4 letters will be in alphabetical order. Then we choose which of 2 places we can lay them down. The last letter is forced.
We might expect there would be a third term of the Inclusion-Exclusion for the completely alphabetical string CDGMT. However this in fact is not what happens. Here's why. The first term accurately counts all the cases where the maximum consecutive alphabetical order is 3. Now consider strings such as *DCGMT* where exactly 4 letters are in alphabetical order. These are counted twice in the first term and taken out once in the second term. So that is right. The final case is the string CDGMT. It is counted three times in the first term and taken out twice in the second term. So we do not need to add it back in by a (false) third term!

NOTE
If you like computer programming this is a good exercise. Here is a way to use Python for this problem (without the correct indenting):

```
def perms01(li):
if len(li)<2:
yield li
else:
for perm in perms01(li[1:]):
for i in range(len(perm)+1):
yield perm[:i] + li[0:1] + perm[i:]

def test(lis):
cnt=0
for k in range(len(lis)-2):
if lis[k]<lis[k+1] and lis[k+1]<lis[k+2]:
cnt=cnt+1
return(cnt)

numb=0

for p in perms01([1,2,3,4,5]):
if test(p)==0:
numb=1+numb

print numb
```

9. Solve:
$$\Big|x-3|x-4|\Big| = |7+2x|$$

   Answer: $x = 5/6$
   By cases.
   Case A. $x \leq 4$. Then we get $4|x-3| = |7+2x|$.
   Case A(i): $x > 3$ and $x > -7/2$. We get $4(x-3) = 7+2x$ or $x = 19/2$, contradiction.
   Case A(ii) $x > 3$ and $x \leq -7/2$. We get $4(x-3) = -7-2x$ or $x = 5/6$. Contradiction.
   Case A(iii) $x \leq 3$ and $x > -7/2$. We get $4(3-x) = 7+2x$ or $x = 5/6$. This works!
   Case A(iv) $x \leq 3$ and $x \leq -7/2$. We get $4(3-x) = -7-2x$ or $x = 19/2$, contradiction.
   Case B $x > 4$ We look at $|-2x+12| = 7+2x$.
   Case B(i) $x \leq 6$. We get $-3x+12 = 7+2x$ or $x = 5/4$. Contradiction.
   Case B(ii) $x > 6$. We get $2x-12 = 7+2x$, already a contradiction.
   Thus the only solution is $x = 5/6$.

10. Suppose n is a positive integer greater than or equal to 10. For such n, define $f(n)$ to be sum of the units digit of n and twice the value of $f(m)$ where m is the integer that remains

when the units digit of n is removed. In case n is a positive integer less than or equal to 9, define $f(n) = n$. Find the value of $f(1145)$.
Answer: 25

$$\begin{aligned} f(1145) &= 5 + 2f(114) \\ &= 5 + 2(4 + 2f(11)) \\ &= 5 + 2(4 + 2(1 + 2f(1))) \\ &= 5 + 2(4 + 2(1 + 2 \cdot 1)) \\ &= 25 \end{aligned}$$

11. Simplify $\left(\sqrt{6}^{\sqrt{12}}\right)^{\sqrt{3}}$.
Answer: 216. .
Using laws of exponents this value is the same as $\sqrt{6}^{\sqrt{36}} = \sqrt{6}^{6} = 6^3 = 216$.

12. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.
Answer: 1 : 05 : 27**.**
Let t be the time in minutes after 1pm that this occurs. Then, in units of degrees, we have $30 + t(1/2) = 0 + 6t$, because the hour hand moves at a rate of $1/2$ of a degree per minute and the minute hand at a rate of 6 degrees per minute. Solving this equation gives $t = 60/11 = 5 + 5/11$. To the nearest second $5/11$ths of a minute is 27 seconds (divide 11 into 300).

13. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 13$.
Answer: $r = 18/5$ **and** $s = 1/5$.
Other answers possible. Please double check your students' answers if they look anywhere close. How to get: Let $r = a/b$ and $s = c/b$. From $a^2 + c^2 = 13b^2$, think of both $13 = 2^2 + 3^2$ and $b^2 = 5^2 = 3^2 + 4^2$ as sum of squares and re-write as:

$$\begin{aligned} a^2 + c^2 &= (2^2 + 3^2)(e^2 + f^2) \\ &= (2^2 + 3^2)(3^2 + 4^2) \\ &= (2 \cdot 3 + 3 \cdot 4)^2 + (2 \cdot 4 - 3 \cdot 3)^2 \\ &= 18^2 + 1^2 \end{aligned}$$

14. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is

heavier. Which ball is different and is it heavier or lighter than the rest?
Answer: 3**, lighter.**
Note this choice is consistent with all three weighings. Also, any different assumption leads to a contradiction. For example, if we assume that is is 7 which is heavier, this is consistent with the first weighing. But then the second weighing would be even, which contradicts what happened.

15. The infinite expression

$$\sqrt{9+\sqrt{9+\sqrt{9+\cdots}}}$$

can be written in the form $\frac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers with no common factor greater than one. Find the value of $a+b+c$.
Answer: 40**.** Let x be the given expression. Then x is found hiding within itself: $x^2 = 9 + x$. Choosing the positive root, the quadratic formula gives $x = (1+\sqrt{37})/2$. Our answer is $1+37+2 = 40$.

Fall 2014 McNabb GDCTM Contest
Geometry Solutions

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?
   Answer: 5.22
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2. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks $4/3$ as fast as Jill. When they meet, how far is Jack from where Jill started?
   Answer: 300 **meters.**
   If Jill walks x meters then Jack walks $4x/3$ in the same time. From $4x/3 + x = 700$ we get $x = 300$, so Jack is 300 meters from where she started when they meet.

3. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?
   Answer: 1122**.**
   Certainly 1122 works since 13 goes into it 86 times with 4 left over, while 19 goes into it 59 times with 1 left over. Any smaller number that works is obtained by taking a 13 by 19 block of soldiers out (since that block can be rotated to fit either way, and no smaller block like that exists). Since $13 \cdot 19 > 122$ we have got the smallest brigade size over 1000.

4. Write down in order from least to greatest (separate by commas) these irrational numbers:
   $$1 + \sqrt{3}, \qquad 2 - \sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$
   Answer: $2 - \sqrt{2}, \sqrt{2}/2, 1 + \sqrt{3}, 2\sqrt{2}$
   The statement $2 - \sqrt{2} < \sqrt{2}/2$ is equivalent to $\sqrt{2} > 4/3$ which is true. Also, the statement $1 + \sqrt{3} < 2\sqrt{2}$ is equivalent to the true statement $2\sqrt{24} + 1 < 11$.

5. Solve:
   $$\Big| x - 3|x - 4| \Big| = |7 + 2x|$$
   Answer: $x = 5/6$
   By cases.
   Case A. $x \le 4$. Then we get $4|x - 3| = |7 + 2x|$.
   Case A(i): $x > 3$ and $x > -7/2$. We get $4(x - 3) = 7 + 2x$ or $x = 19/2$, contradiction.
   Case A(ii) $x > 3$ and $x \le -7/2$. We get $4(x - 3) = -7 - 2x$ or $x = 5/6$. Contradiction.
   Case A(iii) $x \le 3$ and $x > -7/2$. We get $4(3 - x) = 7 + 2x$ or $x = 5/6$. This works!

Case A(iv) $x \leq 3$ and $x \leq -7/2$. We get $4(3-x) = -7-2x$ or $x = 19/2$, contradiction.
Case B $x > 4$ We look at $|-2x+12| = 7+2x$.
Case B(i) $x \leq 6$. We get $-3x+12 = 7+2x$ or $x = 5/4$. Contradiction.
Case B(ii) $x > 6$. We get $2x-12 = 7+2x$, already a contradiction.
Thus the only solution is $x = 5/6$.

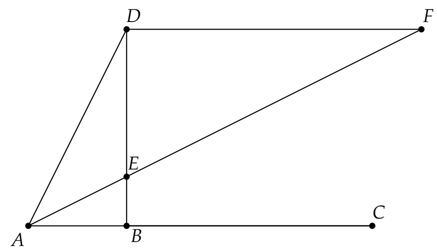
6. A fountain has two basins, one above and one below, each of which has three outlets. The first outlet of the top basin fills the lower basin in one hour, the second in two hours, and the third in three hours. When all three upper outlets are shut, the first outlet of the lower basin empties it in two hours, the second in three hours, and the third in four hours. If all the outlets are opened, how long in hours will it take for the lower basin to fill?
Answer: $4/3$**.**
The rates for filling are $1, 1/2$, and $1/3$; the rates for draining are $1/2, 1/3, 1/4$. The net rate for filling is $1+1/2+1/3-(1/2+1/3+1/4) = 1-1/4 = 3/4$. Thus the lower basin will fill in $4/3$ hours.

7. For m a positive integer, let $g(m)$ be the number of distinct prime factors of m. For example, $g(12) = 2$. Find the value of $g(g(60) \cdot g(91))$.
Answer: 2**.**

$$\begin{aligned} g(g(60) \cdot g(91)) &= g(3 \cdot 2) \\ &= g(6) \\ &= 2 \end{aligned}$$

8. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.
Answer: 1 : 05 : 27**.**
Let t be the time in minutes after 1pm that this occurs. Then, in units of degrees, we have $30 + t(1/2) = 0 + 6t$, because the hour hand moves at a rate of $1/2$ of a degree per minute and the minute hand at a rate of 6 degrees per minute. Solving this equation gives $t = 60/11 = 5 + 5/11$. To the nearest second $5/11$ths of a minute is 27 seconds (divide 11 into 300).

9. In the diagram, segment DB is perpendicular to both DF and AC, and $A-E-F$ are collinear. In addition, $EF = 2AD$. If $\angle FAC$ measures 17° find the angle measure in degrees of $\angle DAB$.



Answer: 51**.**
Let G be the midpoint EF. Then $DG = FG = 5$ and $\angle FDG = 17°$. By the exterior angle theorem $\angle DGA = 17 + 17 = 34°$. But $\triangle ADG$ is isosceles, so $\angle DAF = 34°$. Thus by the Angle Addition Postulate, $\angle DAB = 34 + 17 = 51°$.

10. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?
Answer: 3**, lighter.**
Note this choice is consistent with all three weighings. Also, any different assumption leads to a contradiction. For example, if we assume that is is 7 which is heavier, this is consistent with the first weighing. But then the second weighing would be even, which contradicts what happened.

11. The line $y = mx$ intersects the lines $x + y = 7$ and $x + y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?
Answer: $m = -12/5$ **or** $m = -5/12$.
By inspection: Join point $(-5, 12)$ to $(10, -24)$. Can also reflect this line about $y = -x$.
OR
Let $A = (a, 7 - a)$ and $B = (b, -14 - b)$. Then $39^2 = (a - b)^2 + (21 - (a - b))^2$ Let $u = a - b$. Solve $u^2 + (21 - u)^2 = 39^2$ to get $u = -15$ or $u = 36$. By similar triangles, $2a + b = 0$. Solving the systems gives $(a, b) = (-5, 10)$ and $(a, b) = (12, -24)$, leading to the two values of m given.

12. In right $\triangle ABC$, its inscribed circle meets legs BA and BC at points D and E respectively. If $BD = 3$ and $DA = 11$, find the length of leg BC.

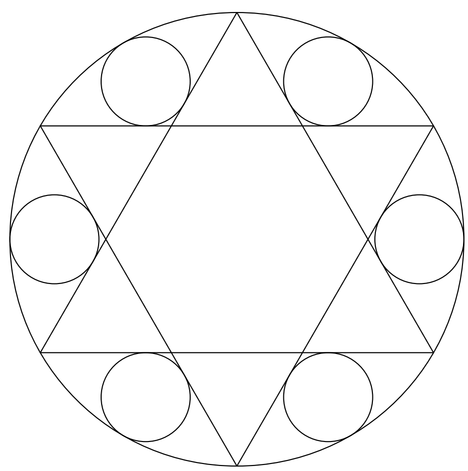
Answer: $BC = 33/4$.
Let $x = EC$. Then by equality of tangents and the Pythagorean Theorem we have

$$(x+11)^2 = (x+3)^2 + 14^2$$

This gives $x = 21/4$. Then $BC = x + 3 = 21/4 + 3 = 33/4$.

13. In $\triangle ABC$, point D lies on segment AB so that $AD/DB = 2/11$ while point E lies on segment BC so that $CE/EB = 3/7$. Let AE and CD intersect at point F. Find the ratio DF/FC.
Answer: $14/39$.
Use Mass Points. Put a mass of 33 at A, of 6 at B, and of 14 at C. Then a mass of 39 at D balances a mass of 14 at C, overall at F. So be the lever principle $DF/FC = 14/39$. Note: Ceva proved his famous theorem using mechanics!

14. A regular six-pointed star, composed of two intersecting equilateral triangles each of side length $6\sqrt{3}$, is inscribed in a circle. Six congruent smaller circles are internally tangent to this circle and externally tangent to the star. Find the radius of the small circles.



Answer: $r = 18\sqrt{3} - 30$.
First note that the radius of the large circle is equal to 6 (because it is 2/3 of the height of one of the equilateral triangles). Consider a 60 degree slice in which one of the smaller circles is centered. Let r be the radius of a small circle. Then the center radius of this slice can be decomposed into three segments so that

$$6 = 2\sqrt{3} + r(2/\sqrt{3}) + r$$

Solving this equation for r gives $r = 18\sqrt{3} - 30$.

15. Find the volume of the tetrahedron with vertices located at $(0,0,0)$, $(1,-2,1)$, $(1,2,-1)$, and $(2,1,-1)$.
Answer: $V = 1/3$.

A nice application of determinants. The volume is one-sixth of the absolute value of the determinant of this matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

OR

Use isosceles triangle with vertices $(0,0,0)$, $(1,2,-1)$, and $(2,1,-1)$ as a base with easy to find area $\sqrt{11}/2$. This triangle lives in plane $x+y+3z=0$. Use distance of point to plane formula to find that the height of the tetrahedron is $2/\sqrt{11}$. Then $V=(1/3)Bh=1/3$ as usual.

Fall 2014 McNabb GDCTM Contest
Algebra Two Solutions

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?
   Answer: 5.22
   The 8 Fiji apples cost $4.68 + 2(.90) = 6.48$ while the 9 Gala apples cost $5.39 + 3(.97) = 8.30$. So the change is $20 - 14.78 = 5.22$.

2. Forty-one erasers are distributed to n students. If at least one student always receives at least 6 erasers no matter how the erasers are distributed, what is the largest possible value of n?
   Answer: 8
   If there are 8 students and we distribute 5 erasers to each, one more eraser must be distributed, so whoever gets that eraser now has 6 erasers. So $n \geq 8$. But if there are nine students it is easy to distribute 5 erasers each to 8 of them and the last eraser to the ninth student. Thus the greatest possible value of n is 8.

3. Simplify
   $$x - y - (3x - 4y) - (x - y - (3x - 4y))$$
   Answer: 0

4. The area of a rectangle is 864. The sum of the length and width is 60. By how much does the length exceed the width?
   Answer: 12
   $(L - W)^2 = (L + W)^2 - 4LW = 60^2 - 4 \cdot 864 = 144.$

5. A regular hexagon is inscribed in a circle of radius one while another regular hexagon is circumcribed about this circle. What is the area of the region enclosed by the two hexagons?
   Answer: $\sqrt{3}/2$.
   The equilateral triangles that comprise the larger hexagon each have side length $2/\sqrt{3}$. The equilateral triangles that comprise the smaller hexagon each have side length 1. Compute: $(6\sqrt{3}/4)((2/\sqrt{3})^2 - 1^2) = \sqrt{3}/2$.

6. How many arrangements of the letters in GDCTM do not have any 3 consecutive letters in alphabetical order? So, for instance, you would count DGCTM but you would not count DGCMT.
   Answer: 70
   Inclusion-Exclusion works well here, with an important catch. Count the opposite case, where 3 or more consecutive letters are in alphabetical order. This works as:
   $$\binom{5}{3} \cdot 3 \cdot 2 - \binom{5}{4} \cdot 2 = 50$$

so that the number we seek is $120 - 50 = 70$.
Note in the first term of Inclusion-Exclusion we are counting places where three consecutive letters are in alphabetical order occur. First we pick which 3 letters will be in alphabetical order. Then we choose which of 3 places we can lay them down. Then we pick the order of the remaining two letters.
In the second term of Inclusion-Exclusion we are counting places where four consecutive letters are in alphabetical order occur. First we pick which 4 letters will be in alphabetical order. Then we choose which of 2 places we can lay them down. The last letter is forced.
We might expect there would be a third term of the Inclusion-Exclusion for the completely alphabetical string CDGMT. However this in fact is not what happens. Here's why. The first term accurately counts all the cases where the maximum consecutive alphabetical order is 3. Now consider strings such as $DCGMT$ where exactly 4 letters are in alphabetical order. These are counted twice in the first term and taken out once in the second term. So that is right. The final case is the string CDGMT. It is counted three times in the first term and taken out twice in the second term. So we do not need to add it back in by a (false) third term!
NOTE
If you like computer programming this is a good exercise. Here is a way to use Python for this problem (without the correct indenting):

```
def perms01(li):
if len(li)<2:
yield li
else:
for perm in perms01(li[1:]):
for i in range(len(perm)+1):
yield perm[:i] + li[0:1] + perm[i:]

def test(lis):
cnt=0
for k in range(len(lis)-2):
if lis[k]<lis[k+1] and lis[k+1]<lis[k+2]:
cnt=cnt+1
return(cnt)

numb=0

for p in perms01([1,2,3,4,5]):
if test(p)==0:
numb=1+numb

print numb
```

7. Solve the system

$$\begin{cases} \dfrac{x}{6} + \dfrac{4}{y} &= 2 \\ \dfrac{18}{x} + \dfrac{y}{2} &= 5 \end{cases}$$

Answer: $(6, 4)$ **and** $(36/5, 5)$.
Multiply the two equations together to obtain $72/(xy) + xy/12 = 5$. Let $u = xy$. Solve the resulting quadratic in u to find $u = 2$ or $u = 3$. The former leads to $(x, y) = (6, 4)$ and the latter to $(x, y) = (36/5, 5)$. Students must list both answers to be correct.

8. Solve:

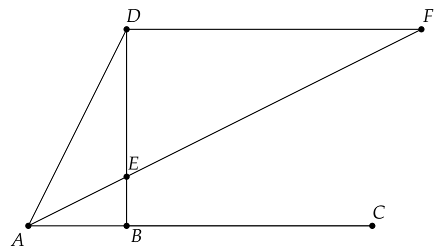
$$\Big| x - 3|x - 4| \Big| = |7 + 2x|$$

Answer: $x = 5/6$
By cases.
Case A. $x \leq 4$. Then we get $4|x - 3| = |7 + 2x|$.
Case A(i): $x > 3$ and $x > -7/2$. We get $4(x - 3) = 7 + 2x$ or $x = 19/2$, contradiction.
Case A(ii) $x > 3$ and $x \leq -7/2$. We get $4(x - 3) = -7 - 2x$ or $x = 5/6$. Contradiction.
Case A(iii) $x \leq 3$ and $x > -7/2$. We get $4(3 - x) = 7 + 2x$ or $x = 5/6$. This works!
Case A(iv) $x \leq 3$ and $x \leq -7/2$. We get $4(3 - x) = -7 - 2x$ or $x = 19/2$, contradiction.
Case B $x > 4$ We look at $| - 2x + 12| = 7 + 2x$.
Case B(i) $x \leq 6$. We get $-3x + 12 = 7 + 2x$ or $x = 5/4$. Contradiction.
Case B(ii) $x > 6$. We get $2x - 12 = 7 + 2x$, already a contradiction.
Thus the only solution is $x = 5/6$.

9. Simplify $\left(\sqrt{6}^{\sqrt{12}}\right)^{\sqrt{3}}$.
Answer: 216. .
Using laws of exponents this value is the same as $\sqrt{6}^{\sqrt{36}} = \sqrt{6}^{6} = 6^3 = 216$.

10. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.
Answer: 1 : 05 : 27**.**
Let t be the time in minutes after 1pm that this occurs. Then, in units of degrees, we have $30 + t(1/2) = 0 + 6t$, because the hour hand moves at a rate of $1/2$ of a degree per minute and the minute hand at a rate of 6 degrees per minute. Solving this equation gives $t = 60/11 = 5 + 5/11$. To the nearest second $5/11$ths of a minute is 27 seconds (divide 11 into 300).

11. In the diagram, segment DB is perpendicular to both DF and AC, and $A - E - F$ are collinear. In addition, $EF = 2AD$. If $\angle FAC$ measures 17° find the angle measure in degrees of $\angle DAB$.



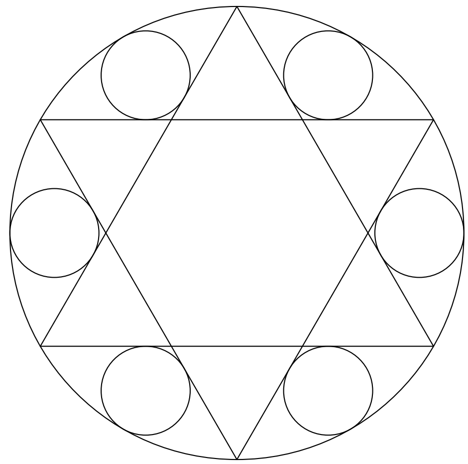
Answer: 51**.**
Let G be the midpoint EF. Then $DG = FG = 5$ and $\angle FDG = 17°$. By the exterior angle theorem $\angle DGA = 17 + 17 = 34°$. But $\triangle ADG$ is isosceles, so $\angle DAF = 34°$. Thus by the Angle Addition Postulate, $\angle DAB = 34 + 17 = 51°$.

12. The line $y = mx$ intersects the lines $x + y = 7$ and $x + y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?
Answer: $m = -12/5$ **or** $m = -5/12$.
By inspection: Join point $(-5, 12)$ to $(10, -24)$. Can also reflect this line about $y = -x$.
OR
Let $A = (a, 7 - a)$ and $B = (b, -14 - b)$. Then $39^2 = (a - b)^2 + (21 - (a - b))^2$ Let $u = a - b$. Solve $u^2 + (21 - u)^2 = 39^2$ to get $u = -15$ or $u = 36$. By similar triangles, $2a + b = 0$. Solving the systems gives $(a, b) = (-5, 10)$ and $(a, b) = (12, -24)$, leading to the two values of m given.

13. Find the value of
$$\sum_{k=1}^{100} i^{k(k+1)/2}$$
Here, i stands for the square root of negative one.
Answer: -2.
The partial sums of $i + i^3 + i^6 + i^10 + \cdots$ follow the pattern
$$i, 0, -1, -2, -2 - i, -2, -1, 0, i, 0, -1, -2, -2 - i, -2, -1, 0, \cdots$$
repeating on a cycle of length eight. Eight goes into 100 with a remainder of 4. So the sum is -2.

14. Find the sum of the real roots of the polynomial $x^4 - 16x^2 - 40x - 25$.
Answer: 4.
Write as difference of squares: $(x^2)^2 - (4x + 5)^2 = 0$ or $(x^2 - 4x - 5)(x^2 + 4x + 5) = 0$. The real roots come from the first quadratic factor.

15. A regular six-pointed star, composed of two intersecting equilateral triangles each of side length $6\sqrt{3}$, is inscribed in a circle. Six congruent smaller circles are internally tangent to this circle and externally tangent to the star. Find the radius of the small circles.



Answer: $r = 18\sqrt{3} - 30$**.**
First note that the radius of the large circle is equal to 6 (because it is 2/3 of the height of one of the equilateral triangles). Consider a 60 degree slice in which one of the smaller circles is centered. Let r be the radius of a small circle. Then the center radius of this slice can be decomposed into three segments so that

$$6 = 2\sqrt{3} + r(2/\sqrt{3}) + r$$

Solving this equation for r gives $r = 18\sqrt{3} - 30$.

Fall 2014 McNabb GDCTM Contest
PreCalculus Solutions

NO Calculators Allowed

1. Express 188 as the sum of two prime numbers.
   Answer: $181 + 7$ **or four others.**
   Others are: $109 + 79$, $157 + 31$, $151 + 37$,and $127 + 61$.

2. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks 4/3 as fast as Jill. When they meet, how far is Jack from where Jill started?
   Answer: 300 **meters.**
   If Jill walks x meters then Jack walks $4x/3$ in the same time. From $4x/3 + x = 700$ we get $x = 300$, so Jack is 300 meters from where she started when they meet.

3. Lincoln Middle School has 324 students and 17 teachers. A field trip for the entire school (students and teachers) to the Kimball Museum is planned. Each of their buses holds at most 28 passengers. Each bus must have at least one teacher on it. What is the minimum number of buses the school requires for this field trip?
   Answer: 13
   To transport 341 people at least 13 buses are needed because $28 \cdot 12 < 341 \leq 28 \cdot 13$. And of course there are more than 13 teachers going.

4. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.
   Answer: acd/b
   In the answer each letter appears once, either as a factor in the numerator or the denominator. Suppose a were to increase while all other letters remain fixed. Then the tea leaves have become weaker so more are needed to fill the thermos's. So a is in the numerator. Suppose b increases. Then the tea leaves have become stronger, so b is in the denominator. If c increases the thermos's are now larger, so c is in the numerator. If d increases more tea must be made, so d is in the numeratory. Thus the answer is acd/b.

5. The pages of the book *Science of Mechanics in the Middle Ages* are numbered from 1 to 711. Considering all the digits needed to print these page numbers starting from page 1, on what page number does the 241st '1'occur?
   Answer: 701
   Pages 1 thru 9 use 1 one. The pages 10 thru 99 use 19 ones. The pages 100 thru 700 use $100 + 60 + 60 = 220$ ones.

6. Recall that $\binom{m}{n}$ stands for the number of ways of choosing n objects from a set of m objects. Name a solution m greater than 2 of the equation

$$\binom{m+2}{3} = 4\binom{m}{2}$$

   Answer: $m = 7$

The stated condition is
$$\frac{(m+2)(m+1)m}{6} = \frac{4m(m-1)}{2}$$
which simplifies to $m^2 - 9m + 14 = (m-2)(m-7) = 0$.

7. Find the value of
$$\sum_{k=1}^{100} i^{k(k+1)/2}$$
Here, i stands for the square root of negative one.
Answer: -2.
The partial sums of $i + i^3 + i^6 + i^10 + \cdots$ follow the pattern
$$i, 0, -1, -2, -2-i, -2, -1, 0, i, 0, -1, -2, -2-i, -2, -1, 0, \cdots$$
repeating on a cycle of length eight. Eight goes into 100 with a remainder of 4. So the sum is -2.

8. Suppose n is a positive integer greater than or equal to 10. For such n, define $f(n)$ to be sum of the units digit of n and twice the value of $f(m)$ where m is the integer that remains when the units digit of n is removed. In case n is a positive integer less than or equal to 9, define $f(n) = n$. Find the value of $f(1145)$.
Answer: 25
$$\begin{aligned} f(1145) &= 5 + 2f(114) \\ &= 5 + 2(4 + 2f(11)) \\ &= 5 + 2(4 + 2(1 + 2f(1))) \\ &= 5 + 2(4 + 2(1 + 2 \cdot 1)) \\ &= 25 \end{aligned}$$

9. Let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x. Write the solution of the equation
$$\lfloor 2x \rfloor = \lfloor 3x \rfloor$$
using interval notation.
Answer: $[-2/3, -1/2) \cup [-1/3, 1/3) \cup [1/2, 2/3)$.
A value x solves this equation iff there exists an integer n so that both
$$\begin{aligned} n \le 2x < n+1 \\ n \le 3x < n+1 \end{aligned}$$
are true. This means both
$$\begin{aligned} n/2 \le 2x < n/2 + 1/2 \\ n/3 \le 3x < n/3 + 1/3 \end{aligned}$$
are true. The intervals $[n/2, n/2 + 1/2)$ and $[n/3, n/3 + 1/3)$ overlap only when $n = -2, -1, 0, 1$. These intersections form the three intervals of the solution given above (two of them fit back to back and so form a single interval in fact).

10. Define recursively the function $a(m,n)$:

$$\begin{aligned} a(m,n) &= a(m-1, a(m,n-1)) \\ a(0,n) &= n+1 \\ a(m,0) &= a(m-1,1) \end{aligned}$$

Find the value of $a(2,2)$.
Answer: $a(2,2) = 7$**.**
Recursively build up from smaller values. Key values along the way are: $a(2,1) = 5$, $a(2,0) = 3$, $a(1,5) = 7$, $a(1,0) = 2$, $a(1,3) = 5$, $a(1,2) = 4$, and $a(1,1) = 3$. This is a famous function in the history of logic, the Ackermann function.

11. Find the volume of the tetrahedron with vertices located at $(0,0,0)$, $(1,-2,1)$, $(1,2,-1)$, and $(2,1,-1)$.
Answer: $V = 1/3$**.**
A nice application of determinants. The volume is one-sixth of the absolute value of the determinant of this matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

OR
Use isosceles triangle with vertices $(0,0,0)$, $(1,2,-1)$, and $(2,1,-1)$ as a base with easy to find area $\sqrt{11}/2$. This triangle lives in plane $x+y+3z=0$. Use distance of point to plane formula to find that the height of the tetrahedron is $2/\sqrt{11}$. Then $V = (1/3)Bh = 1/3$ as usual.

12. The line $y = mx$ intersects the lines $x+y = 7$ and $x+y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?
Answer: $m = -12/5$ **or** $m = -5/12$.
By inspection: Join point $(-5,12)$ to $(10,-24)$. Can also reflect this line about $y = -x$.
OR
Let $A = (a, 7-a)$ and $B = (b, -14-b)$. Then $39^2 = (a-b)^2 + (21-(a-b))^2$ Let $u = a-b$. Solve $u^2 + (21-u)^2 = 39^2$ to get $u = -15$ or $u = 36$. By similar triangles, $2a+b = 0$. Solving the systems gives $(a,b) = (-5,10)$ and $(a,b) = (12,-24)$, leading to the two values of m given.

13. In $\triangle ABC$ point D lies on side BC so that AD bisects angle BAC. If $AD = 13$, $DC = 37$, and $AC = 40$, find the length of AB.
Answer: $6760/231$.
Let $x = AB$ and $y = BD$. By the Angle Bisector Theorem $y = (37/40)x$. By the Law of Sines,

$$\frac{(37/40)x + 37}{2 \cdot (12/13)(5/13)} = \frac{x}{12/37}$$

This simplifies to $169(x+40) = 400x$. Thus $AB = x = 6760/231$.

14. Let $f(x)$ be a function which satisfies for all x and y the relation

$$f(x) \cdot f(y) - f(xy) = x + y$$

Determine the function $f(x)$.
Answer: $f(x) = x + 1$.
Let $y = 0$. Then $f(0) \cdot f(x) = x + f(0)$. Let $x = 0$ and $y = 0$. Then $f^2(0) - f(0) = 0$. So either $f(0) = 0$ or $f(0) = 1$. But if $f(0) = 0$ then from what we just did it would follow that $0 = x + 0$, for all x, which is impossible. Thus $f(0) = 1$ and $f(x) = x + 1$.

15. In how many ways can 4 different rings be placed on the four fingers of the right hand? Here the order of the rings on a given finger matters and each finger can accomodate all four rings.
Answer: 840
Notation: $a - b - c - d$ means one finger gets a rings, another b rings, etc..
Case I. $4 - 0 - 0 - 0$ can occur in $4 \cdot 4! = 96$ ways (as in 4 choices as to which finger gets all four rings, and then the number of ways to permute the 4 rings on that finger.).
Case II. $3 - 1 - 0 - 0$ can occur in $4 \cdot 4 \cdot 3! \cdot 3 = 288$ ways. The first 4 if for choosing which 3 rings go together. The second 4 is for choosing which finger to put them on. The 3! is for permuting those three rings on that finger. The 3 is for choosing which of the remaining three fingers to place the final ring on.
Case III. $2 - 2 - 0 - 0$ can occur in $3\dot{4} \cdot 2 \cdot 3 \cdot 2 = 144$ ways. The first 3 is for choosing who goes with ring A. (Label the rings say A, B, C, D.) The four is for choosing which finger A and its match go on. The two is for permuting A and its match. The 3 is for choosing which remaining finger to put the last two rings on. The two is for permuting those last two rings.
Case IV. $2 - 1 - 1 - 0$ can occur in $6 \cdot 4 \cdot 2 \cdot 3 \cdot 2 = 288$ ways. The six is for choosing which two rings will be paired. The four is for choosing which finger to put the pair on. The two is for permuting that pair. There are now two rings left, call them C and D. The three is for choosing which of three fingers to put C on. The two is for choosing which of two fingers to put D on.
Case V. $1 - 1 - 1 - 1$ can occur in 4! ways.
And $96 + 288 + 144 + 288 + 24 = 840$.
OR
The distribution of the number of rings assigned to each finger can be counted by stars and bars as $\binom{7}{3} = 35$. Then multiply this by $4! = 24$ as the four rings are distinct. Hence, $35 \cdot 24 = 840$.

Fall 2014 McNabb GDCTM Contest
Calculus Solutions

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. A number of students pitch in to buy a gift for their teacher. If each pays 8 dollars, the total collected would be too great by 3 dollars. If each pays 7 dollars, the total collected would be too little by 4 dollars. How much does the gift cost?
   Answer: 53**.**
   The difference of the over and under is 7 dollars so there must be 7 students. Then $56 - 3 = 53$ is the amount of the gift.

2. Write down in order from least to greatest (separate by commas) these irrational numbers:

$$1 + \sqrt{3}, \qquad 2 - \sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

   Answer: $2 - \sqrt{2}, \sqrt{2}/2, 1 + \sqrt{3}, 2\sqrt{2}$
   The statement $2 - \sqrt{2} < \sqrt{2}/2$ is equivalent to $\sqrt{2} > 4/3$ which is true. Also, the statement $1 + \sqrt{3} < 2\sqrt{2}$ is equivalent to the true statement $2\sqrt{24} + 1 < 11$.

3. Solve the system

$$\begin{cases} \dfrac{x}{6} + \dfrac{4}{y} & = 2 \\ \dfrac{18}{x} + \dfrac{y}{2} & = 5 \end{cases}$$

   Answer: $(6, 4)$ **and** $(36/5, 5)$.
   Multiply the two equations together to obtain $72/(xy) + xy/12 = 5$. Let $u = xy$. Solve the resulting quadratic in u to find $u = 2$ or $u = 3$. The former leads to $(x, y) = (6, 4)$ and the latter to $(x, y) = (36/5, 5)$. Students must list both answers to be correct.

4. In right $\triangle ABC$, its inscribed circle meets legs BA and BC at points D and E respectively. If $BD = 3$ and $DA = 11$, find the length of leg BC.
   Answer: $BC = 33/4$**.**
   Let $x = EC$. Then by equality of tangents and the Pythagorean Theorem we have

$$(x + 11)^2 = (x + 3)^2 + 14^2$$

   This gives $x = 21/4$. Then $BC = x + 3 = 21/4 + 3 = 33/4$.

5. Find one positive value of b so that $x = \tan^{-1} b$ solves the equation

$$\tan^2(2x) + \tan^2(x) = 10$$

   Answer: $b = \sqrt{2}$ **OR** $b = \sqrt{5 \pm 2\sqrt{5}}$**.**

We have, using the double angle formula for tangents followed by algebra:

$$\begin{aligned}\left(\frac{2\tan x}{1-\tan^2 x}\right)^2 &= 10-\tan^2 x\\ 4\tan^2 x &= (10-\tan^2 x)(1-\tan^2 x)^2\\ 4b^2 &= (10-b^2)(1-b^2)^2\\ 4b^2 &= 10-21b^2+12b^4-b^6\\ b^6-12b^4+35b^2-10 &= 0\\ (b^2-2)(b^4-10b^2+5) &= 0\end{aligned}$$

So $b=\sqrt{2}$ works as do $b=\sqrt{5\pm 2\sqrt{5}}$.

6. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1,4,7,10$ are put in the left pan and balls $3,6,9,12$ are put in the right pan, the left pan is heavier. When balls $3,6,9,10$ are put in the left pan and balls $2,5,8,12$ are put in the right pan, the left pan is lighter. When balls $3,4,8,12$ are put in the left pan and balls $2,6,7,11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?
   Answer: 3**, lighter.**
   Note this choice is consistent with all three weighings. Also, any different assumption leads to a contradiction. For example, if we assume that is is 7 which is heavier, this is consistent with the first weighing. But then the second weighing would be even, which contradicts what happened.

7. Let the function y satisfy
$$\frac{dy}{dx}=y^2(y-2)(y+2)$$
   If $y(0)=-1/5$ find the limit as $x\to\infty$ of $y(x)$.
   Answer: -2**.**
   Plot dy/dx versus y. Starting at $y=-1/5$, note $dy/dx<0$ so as x increases y must decrease. This continues happening as dy/dx continues negative while $-2<y<-1/5$. Thus y is driven to $y=-2$ as x goes to ∞. For this solution it never happens that $y=-2$ because this is an equilibrium solution of the differential equation and, by uniqueness, cannot be crossed.

8. Let
$$f(x)=\left(\sum_{k=0}^{100}x^k\right)\cdot\left(\sum_{k=0}^{100}(-1)^k x^k\right)$$
   Determine the value of $f'(1)$.
   Answer: 10100**.**

By the product rule,

$$\begin{aligned} f'(1) &= (1+2+3+\cdots+100)(1) + (101)(-1+2-3+4-\cdots-99+100) \\ &= 5050 + 101\cdot 50 \\ &= 10100 \end{aligned}$$

9. Let f be differentiable for all x. Let $f(1) = 1/2$. Suppose for all $x \neq 0$ that $f(x) = f(1/x)$. Let $g(x) = x^4 f(x)$. Find the value of $g'(1)$.
Answer: 2.
Note $f'(x) = f'(1/x)(-1/x^2)$ so $f'(1) = f'(1)\cdot(-1)$. Thus $f'(1) = 0$. By the Product Rule, $g'(x) = 4x^3 f(x) + x^4 f'(x)$. And $g'(1) = 4f(1) + 1(f'(1)) = 4(1/2) + 0 = 2$.

10. Let a be a positive constant. If the minimum value of $f(x) = e^x + ae^{-x}$ is $3a$, find the value of a.
Answer: $a = 4/9$.
Since $a > 0$, f is concave up everywhere. Thus its only critical point is where it global minimum value occurs. Set $f' = 0$ to find $x = \ln a/2$. Now $f(\ln a/2) = 2\sqrt{a} = 3a$ or $\sqrt{a} = 2/3$. Thus $a = 4/9$.

11. Determine

$$\lim_{x\to 0} (\cos x)^{\cot^2 x}$$

Answer: $e^{-1/2}$.
Take the ln and find $\lim_{x\to 0} \ln\cos x/\tan^2 x$ using L'Hospital's rule to be $-1/2$. Log out to find answer.

12. The infinite expression

$$\sqrt{9+\sqrt{9+\sqrt{9+\cdots}}}$$

can be written in the form $\dfrac{a+\sqrt{b}}{c}$ where a, b, and c are positive integers with no common factor greater than one. Find the value of $a+b+c$.
Answer: 40. Let x be the given expression. Then x is found hiding within itself: $x^2 = 9 + x$. Choosing the positive root, the quadratic formula gives $x = (1+\sqrt{37})/2$. Our answer is $1 + 37 + 2 = 40$.

13. Find the equation of a line which is tangent to $y = x^4 - 8x^2 + 3x + 5$ at two distinct points.
Answer: $y = 3x - 11$.
Note that $y = x^4 - 8x^2 + 16 = (x^2-4)^2$ has a graph which is a perfect "W" shape with $y = 0$ tangent to the two distinct minimum points. Thus $y = x^4 - 8x^2 + 16 + (3x - 11)$ has a bitangent $y = 3x - 11$.

14. Let

$$f(x) = \begin{cases} e^{3x-(1/x^2)} + 2e^{3x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

Determine the value of $f'(0)$.
Answer: $f'(0) = 6$**.**
Let $f(x) = e^{3x}g(x)$. Then by the product rule

$$f'(x) = 3e^{3x}g(x) + e^{3x}g'(x)$$

Using the definition of derivative one can prove that $g'(0) = 0$. So $f'(0) = 3 \cdot 1 \cdot 2 + 1 \cdot 0 = 6$.

15. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 13$.
Answer: $r = 18/5$ **and** $s = 1/5$.
Other answers possible. Please double check your students' answers if they look anywhere close. How to get: Let $r = a/b$ and $s = c/b$. From $a^2 + c^2 = 13b^2$, think of both $13 = 2^2 + 3^2$ and $b^2 = 5^2 = 3^2 + 4^2$ as sum of squares and re-write as:

$$\begin{aligned} a^2 + c^2 &= (2^2 + 3^2)(e^2 + f^2) \\ &= (2^2 + 3^2)(3^2 + 4^2) \\ &= (2 \cdot 3 + 3 \cdot 4)^2 + (2 \cdot 4 - 3 \cdot 3)^2 \\ &= 18^2 + 1^2 \end{aligned}$$